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ROTATIONAL CONFORMAL TRANSFORMATIONS OF THE LOBACHEVSKIĬ PLANE

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In a two-dimensional Riemannian space (M^2, g) of Gaussian curvature $K \neq 0$ a curve whose geodesic curvature satisfies the condition $k_g = cK$, $c = \text{const}$, is an isoperimetric rotation extremal (IRE). A transformation of the space, or of its region, which sends geodesic curves to IREs is said to be rotational. Moreover, a local infinitesimal transformation $\widetilde{x}^h = x^h + \varepsilon X^h(x)$ sending each geodesic curve to a curve which is an IRE up to ε^2, is also called an infinitesimal rotational transformation (see [1]–[3]). The geodesic transformations are trivial rotational transformations.

Suppose that a vector field X generates an infinitesimal conformal transformation, i.e.,

$$L_X g_{ij} = \nabla_i X_j + \nabla_j X_i = 2\varphi g_{ij}. \tag{1}$$

This transformation is rotational if and only if the corresponding conformality function $\varphi(x)$ generates a special concircular covector field $\varphi_i = \partial_i \varphi$ such that

$$\nabla_j \varphi_i = \varphi_i \partial_j \ln |K| + (C - \varphi) K g_{ij}, \quad C \text{ is a constant.} \tag{2}$$

If $\varphi = C$, then the transformation is a homothety, and, consequently, is geodesic (i.e., is a trivial rotational transformation). If φ is not constant, then (M^2, g) is locally isometric to a surface of revolution $x = r\cos v$, $y = r\sin v$, $z = f(r)$,

$$(g_{ij}) = \operatorname{diag}\left(1 + \left(\frac{df}{dr}\right)^2, \; r^2\right).$$

If we integrate equations (1), (2) on the surface of revolution, we then obtain

$$C = 0, \quad \varphi = \frac{c_1}{\sqrt{1 + \left(\frac{df}{dr}\right)^2}}, \quad X^h = \left(\frac{c_1 r}{\sqrt{1 + \left(\frac{df}{dr}\right)^2}}, \; c_2\right), \quad c_1, c_2 \text{ are constants.}$$

In particular, on the pseudosphere of radius R, which is a local model of the Lobachevskiĭ plane, we have $(g_{ij}) = \begin{pmatrix} \frac{R^2}{r^2} & 0 \\ 0 & r^2 \end{pmatrix}$, and

$$\frac{df}{dr} = \frac{\sqrt{R^2 - r^2}}{r}, \quad X^h = \left(\frac{c_1 r^2}{R}, \; c_2\right), \quad K = -\frac{1}{R^2}.$$

The vector fields $\underset{1}{X}^h = \left(\frac{r^2}{R}, 0\right)$, $\underset{2}{X}^h = (0, 1)$ constitute a basis of the operators of the two-parameter group of local transformations of the pseudosphere: $\widetilde{v} = v + \tau$, $\widetilde{r} = \frac{Rr}{R - rt}$. These transformations