Russian Mathematics (Iz. VUZ) *Izvestiya VUZ. Matematika*
Vol. 47, No. 1, pp.58–71, 2003 *UDC 517.956*

WEIGHT ESTIMATES OF SOLUTION OF THE DIRICHLET PROBLEM WITH ANISOTROPIC DEGENERATION ON A PART OF BOUNDARY

M.R. Timerbayev

1. Introduction

In this article we investigate the solvability of the following model problem in various weight classes of functions:

$$-D_m(x_m^\alpha a_{mm}(x)D_m u(x)) - x_m^\beta \sum_{i,j<m} D_i(a_{ij}(x)D_j u(x)) = f(x) \quad \text{for} \quad x \in Q, \tag{1}$$

$$u|_{\partial Q} = 0. \tag{2}$$

Here $Q = \Omega \times (0,1)$ is a tube domain in R^m, Ω is a regular domain in R^{m-1}, $(a_{ij}(x))$ is a symmetrical positive definite matrix for $x \in \overline{Q}$ with sufficiently smooth coefficients (for the conditions upon the coefficients see in Sections 3, 4), α, β are real numbers satisfying the condition $\alpha < \min(1,\, \beta + 2)$. This problem arises, for example, in the description of a stationary distribution of temperature in a heat-conducting medium with essentially varying (for $\alpha \neq \beta$) conducting properties in a neighborhood of a part of the boundary $\Gamma = \partial Q \bigcap \{x_m = 0\}$ in the direction of the normal to Γ and along the tangent directions. For positive values of α or β, a closure of conduction on Γ occurs, which leads to appearance of a boundary layer in the neighborhood of degeneration of the coefficients; negative values simulate the superconductivity of the medium. Thus, for $\alpha^2 + \beta^2 \neq 0$, the solution has a singularity as $x_m \to 0$, which must be taken into account in numerical solving.

Usually, for estimation of solutions of problems with degeneration of coefficients, for the proper solution weight norms in the Sobolev spaces are applied (see, e. g., [1]–[3]). In this case, it is not possible to obtain two-sided a priori estimates on the class of regular right sides, because the specificity of the problem is such that even for smooth right sides the solution has a boundary layer. Our approach consists of the use of norms of the weight Sobolev spaces for solution divided by the weight $x_m^{1-\alpha}$ which defines this boundary layer in the neighborhood of the degeneration of coefficients. It is far to be evident that the result of such division will be a smooth function adequate to the smoothness of the right side. A priori estimates obtained in this article show that, indeed, the solution of (1), (2) in the neighborhood of Γ can be represented in the form $u(x) = x_m^{1-\alpha}\varphi(x)$ (or in the form $u(x) = x_m^{-\alpha}\varphi(x)$), where $\varphi(x)$ is sufficiently smooth (which depends on $f(x)$). On the base of this representation, in [4] for problem (1), (2) with an isotropic degeneration ($\alpha = \beta$) the author suggested a scheme of approximation, which coincides by its effectiveness with the usual scheme of the finite element method for regular problems. In the same source the main result of this article was announced. Let us note also that estimates obtained in this article are new also for the regular case $\alpha = \beta = 0$.

The work was supported by the Russian Foundation for Basic Research (code of project 01-01-00616).