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On Stability of a Differential Equation with Aftereffect

T. L. Sabatulina* **and V. V. Malygina****

Perm National Research Polytechnic University, Komsomol'skii pr. 29, Perm, 614990 Russia

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Abstract—We study a linear differential equation with bounded aftereffect and establish conditions for the exponential and uniform stability of its solution in the form of domains in the parameter space. We construct examples that show the exactness of boundaries of stability domains for two classes of functional differential equations with concentrated and distributed delays. Along with classical methods of the functional analysis and function theory, we also use the test equations method.

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Dedicated to Professor N. V. Azbelev
on the occasion of his 90th birthday

In this paper we study the stability of a linear differential equation with aftereffect. We consider a rather general form of this equation, so one can hardly establish necessary and sufficient stability conditions for it. Therefore we restrict ourselves to establishing sufficient conditions, mainly, those ones which allow us to demonstrate the significance of constraints and the exactness of constants (that appear in these conditions) and boundaries of stability domains.

We use the following denotations: $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}_+ = [0, +\infty)$, $\Delta = \{(t, s) \in \mathbb{R}^2_+ : t \geqslant s\}$, C is the space of continuous on $\mathbb{R}_+$ functions with the norm $\|x\|_C = \sup_{t\in\mathbb{R}_+} |x(t)|$, L_1 is the space of summable on $\mathbb{R}_+$ functions with the norm $\|x\|_1 = \int\limits_0^\infty |x(t)|\,dt$, L_∞ is the space of measurable and bounded in essence on $\mathbb{R}_+$ functions with the norm $\|x\|_\infty = \operatorname*{vrai\,sup}\limits_{t\in\mathbb{R}_+} |x(t)|$, $L_{\rm loc}$, and $D_{\rm loc}$ is the space of functions summable and absolutely continuous on each finite segment.

1. THE OBJECT OF THE STUDY. THE MAIN DEFINITIONS

Consider the differential equation

$$\dot{x}(t) + ax(t) + (Tx)(t) = f(t), \quad t \in \mathbb{R}_+, \tag{1}$$

where $a \in \mathbb{R}$, $f \in L_{\rm loc}$, and T is a linear operator acting from the space C into that $L_{\rm loc}$.

If a continuous function x is a solution to Eq. (1), then in view of the equation and properties of the operator T we obtain $\dot{x} \in L_{\rm loc}$, i.e., $x \in D_{\rm loc}$. That is why we usually understand a *solution* to Eq. (1) ([1], P. 9) as a locally absolutely continuous function satisfying this equation almost everywhere.

Recall that an operator T is said to be *regular* ([2], P. 31), if it is representable as the difference of two isotonic operators.

Following [3], an operator T is said to be a *Volterra* one, if for each $c > 0$ the equality $(Tx)(t) = 0$ takes place almost everywhere on $[0, c]$ for all $x \in C$ such that $x(t) = 0$ on $[0, c]$.

*E-mail: tsabatulina@gmail.com.
**E-mail: mavera@list.ru.