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On Classification of von Neumann Algebras in a Space with Conjugation

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Abstract—In this paper for the first time we show that in the complex Hilbert space with the conjugation operator a classification of von Neumann algebras is possible. Similar classification is known for Krein spaces. Projectors (idempotents) often serve as elements of quantum logic. In operator theories projectors play the role of elements from which bounded operators are constructed. For one special case we show that for any projector from von Neumann algebra which acts in a separable Hilbert space one can always find conjugation operator J adjoined to this algebra for which the projector is self-adjoint.

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1. Definitions and some general properties. Let H be a complex Hilbert space with a scalar product (s. p.) $(\cdot,\cdot)$ and S be a unit sphere in H. An operator J is called a *conjugation* operator in H ([1], § 50), if

$$1)\ J^2 = I, \qquad 2)\ (Jx, Jy) = (y, x)\ \ \forall x, y \in H.$$

From 1) and 2) it follows that $J(\lambda x + \beta y) = \overline{\lambda} Jx + \overline{\beta} Jy$ for all $\lambda, \beta \in \mathbb{C}$. A vector $x \in H$ is called J-*real*, if $Jx = x$. The vectors $x_{\Re} := \frac{1}{2}(x + Jx)$ and $x_{\Im} := \frac{1}{2i}(x - Jx) = -\frac{1}{2}(ix + Jix)$ are J-real for all $x \in H$. Let $H_{J\Re}$ be the set of all J-real vectors. It is clear that $(\alpha x)_{\Re} = (\Re\alpha)x$, $(\alpha x)_{\Im} = (\Im\alpha)x$, $(x, y) = (Jx, Jy) = (y, x)$ for all $x, y \in H_{J\Re}$ and $H_{J\Re}$ is a real Hilbert space with respect to the product $(\cdot,\cdot)$. Besides,

$$H_{J\Re} \cap iH_{J\Re} = \{0\},\ \ H = H_{J\Re} + iH_{J\Re}. \tag{1}$$

Remark 1. There is a one-to-one correspondence between real Hilbert subspaces of space H satisfying (1) and conjugation operators in H.

Let $B(H)$ be the set of all bounded operators in H. Let us denote by $B(H)^{\mathrm{Id}}$ ($B(H)^{\mathrm{or}}$) the set of all bounded (orthogonal (self-conjugate)) projectors from $B(H)$. Let us introduce a partial order in $B(H)^{\mathrm{Id}}$ assuming $Q \le P \Leftrightarrow PQ = QP = P$. Let $P \perp Q \Leftrightarrow PQ = QP = 0$ and $P^{\perp} := I - P$ for all $P, Q \in B(H)^{\mathrm{Id}}$. By E_p we denote the orthogonal projector on the subspace PH, i.e., $E_p = \mathrm{ran}(P)$ for any $P \in B(H)^{\mathrm{Id}}$, and P_{or} stands for an orthogonal projector on $PH \cap P^*H$. It is obvious that $P_{\mathrm{or}} \le E_p$.

Operator $A \in B(H)$ is called J-*real*, if $JAJ = A$. Let us note that A is J-real if and only if $AH_{J\Re} \subseteq H_{J\Re}$. The set of all J-real operators forms a real algebra. It is easy to see that an operator A is J-real if and only if A^* is a J-real operator. Let $\langle x, y\rangle := (Jx, y)$, $x, y \in H$. Operator $B^{\#} \in B(H)$ is called J-*conjugate* to $B \in B(H)$, if $\langle Bx, y\rangle = \langle x, B^{\#}y\rangle$, $x, y \in H$. It is obvious that $B^{\#} = JB^*J$.

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