Russian Mathematics (Iz. VUZ) *Izvestiya VUZ. Matematika*
Vol. 42, No. 8, pp.20–26, 1998 *UDC 514.762*

AN INTERPRETATION OF THE TORSION TENSOR OF A CONNECTION

A.A. Grigoryants

1. Introduction

As it is known, among the invariants of a connection on a manifold, the curvature and torsion tensors are the most important. The classical geometric interpretations of these tensors are well-known (see, e. g., [1], pp. 417–420, 494–507). However, in contrast to the case of curvature tensor, the geometric interpretation of the torsion tensor is considered by some authors as not quite satisfactory (see, e. g., [2], p. 50). In the present article a new geometric interpretation of torsion tensor is suggested, which (in the author's opinion) in contrast to the classical one, is invariant and geometrically clear. As far as the author knows, the suggested approach is indeed new (see the surveys [3] and [4]).

Let ∇ be a connection on a manifold M (for the sake of brevity, we use the term "connection" instead of "linear connection") and ∇^* be the conjugate connection in the cotangent bundle T^*M. Let H^* be the corresponding distribution on T^*M (in what follows we often will not distinguish in terminology between a connection and the corresponding distribution). Consider the distribution $H^{*\perp}$ which is orthogonal to H^* with respect to the canonical symplectic structure on the cotangent bundle. We shall prove that $H^{*\perp}$ is also a connection. One of the basic results of the present article is the following: *let $\nabla^\perp$ be the connection operator of a connection conjugate to $H^{*\perp}$, then $T = \nabla - \nabla^\perp$ is the torsion tensor of the connection ∇.*

As a consequence, it follows that the torsion tensor vanishes if and only if H^* is a Lagrangian distribution, i. e., H^*_p is a Lagrangian subspace of the symplectic space T_pT^*M, where $p \in T^*M$. Thus, one can say that the torsion tensor of a connection ∇ is "a measure of non-Lagrangeness" of the conjugate connection ∇^*. The exact definitions will be given in Section 3.

It is not surprising that, among various invariant definitions, the definition of a linear connection in a vector bundle as a distribution on the total space does not find favor. This can be explained by non-invariance of the sense of the following phrase: "a horizontal distribution which linearly depends on the fiber coordinates". However, this definition has an advantage owing to its geometric clarity. Since our approach bases just on this definition, we shall formulate it in an invariant way (see Section 2).

The connection $\nabla^\perp$ can be constructed immediately in terms of ∇, i. e., without invoking T^*M. The connections ∇ and $\nabla^\perp$ are related by means of the canonical involution of the second tangent bundle. In terms of connection operators, this relation can be expressed as follows: $\nabla^\perp_X Y = \nabla_Y X + [X, Y]$. There is a similar relation between the connection operators of the connections $H^{*\perp}$ and H^*, which can be denoted by $\nabla^{*\perp}$ and ∇^*, respectively. Namely, $\langle \nabla^{*\perp}_X \alpha, Y\rangle = \langle \nabla^*_Y \alpha, X\rangle + \langle d\alpha, X \wedge Y\rangle$. Our terminology and notation are close to those adopted in [5]. We shall denote by E a vector bundle $\pi : E \to M$. We also assume that $n = \dim E$ and $m = \dim M$. The module of sections