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ON THE REDUCTION OF DEGENERATE SYSTEMS OF INTEGRAL EQUATIONS OF THE VOLTERRA TYPE TO SYSTEMS OF THE SECOND KIND

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1. Statement of the problem

We consider the following system of integral equations of the Volterra type:

$$W(x(t)) = Ax(t) + \int_0^t K(t-\tau)x(\tau)d\tau = f(t), \quad t \in [0,1], \tag{1}$$

where A, $K(u)$ are $n \times n$-matrices, $f(t)$ is a sufficiently smooth given function, and $x(t)$ is a continuous n-dimensional vector function which is to be found; $u = t - \tau$.

Systems of the form (1) with $A = 0$ are usually called the systems of the first kind, while the systems with $\det A \neq 0$ — of the second kind (see [1]).

In the present article we consider the systems of the form (1) with $A \neq 0$ and $\det A \neq 0$. Such systems will be called degenerate. The input data of the initial problem are assumed to have the smoothness required in the subsequent reasoning. Our purpose is to discover the properties of the matrices A, $K(u)$ and the vector function $f(t)$ in (1) so that

a) the initial problem have a unique solution in the class of continuous functions;

b) a linear differential operator of the order m exist

$$\mathcal{P}_m = \sum_{i=0}^{m} (d/dt)^i p_i = \sum_{i=0}^{m} p_i (d/dt)^i, \tag{2}$$

where p_i are $n \times n$-matrices, such that the system

$$\mathcal{P}_m\Big(Ax(t) + \int_0^t K(t-\tau)x(\tau)d\tau\Big) = \overline{A}x(t) + \int_0^t \overline{K}(t-\tau)x(\tau)d\tau = \mathcal{P}_m f(t), \quad t \in [0,1], \tag{3}$$

is of the second kind.

Observe that such systems of the second kind for $K(u), f(t) \in C$ have a unique continuous solution (see [1]). Let us illustrate the latter by the following example:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x(t) + \int_0^t \begin{pmatrix} 1 & 0 \\ 0 & a(t-\tau) \end{pmatrix} x(\tau)d\tau = \begin{pmatrix} 1 \\ f(t) \end{pmatrix}. \tag{4}$$

We immediately verify that the problem has the unique continuous solution $x(t) = (\exp(-t), f''(t)/a)^\top$ if and only if

$$f(t) \in C^2, \quad a \neq 0, \quad f(0) = f'(0) = 0.$$

If $f(0) \neq 0$, or $f'(0) \neq 0$, or $a = 0$ and $f(t) \not\equiv 0$, then such a system has no continuous solution.