ISSN 1066-369X, Russian Mathematics (Iz. VUZ), 2009, Vol. 53, No. 6, pp. 49–52. © Allerton Press, Inc., 2009.
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Mappings Connected with the Gradient of Conformal Radius

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Received June 18, 2007

Abstract—In this paper we prove the following conformity criterion for the gradient of conformal radius $\nabla R(D, z)$ of a convex domain D: the boundary ∂D has to be a circumference. We calculate coefficients $K(r)$ for $K(r)$-quasiconformal mappings $\nabla R(D(r), z)$, $D(r) \subset D$, $0 < r < 1$, and complete the results obtained by F. G. Avkhadiev and K.-J. Wirths for the structure of boundary elements of quasiconformal mappings of the domain D.

DOI: 10.3103/S1066369X09060085

Key words and phrases: *conformal radius, gradient of conformal radius, K-quasiconformal mapping, Beltrami equation.*

The first studies of diffeomorphic mappings of a domain D by means of the gradient function

$$\nabla R(D, z) = 2\frac{\partial R}{\partial \bar{z}}, \tag{1}$$

where $R(D, z)$ is the conformal radius of the domain D at a point z, are performed by F. G. Avkhadiev and K.-J. Wirths [1, 2]. In this paper we supplement the results of the mentioned two publications with a theorem about the construction of boundary curves by means of rolling.

The main result of this paper is the following conformity criterion for the gradient of conformal radius (1): the boundary of the domain D has to be a circumference. For any convex domain $D = f(E)$ we introduce the coefficient of quasiconformality $k_f(r)$. It characterizes the K-quasiconformal mapping of the domain $D(r) = f(rE)$, $K = \frac{1+k_f(r)}{1-k_f(r)}$. We evaluate this coefficient for several domains (in particular, for domains with the degenerate gradient [2]) and consider the calculation of the coefficient

$$k(r) = \sup_{f \in S^0} k_f(r)$$

for the whole class S^0 of convex mappings.

I. As noted in papers [1, 2], function (1) in a polygonal domain D is rather distinctive. It defines a diffeomorphic mapping $w(z) = \nabla R(D, z)$ of the domain D, but this diffeomorphism degenerates on the boundary ∂D. Every rectilinear side $l_k \subset \partial D$ goes into a point A_k of the circumference $|w| = 2$, while every angular vertex $a_k \in \partial D$ corresponds to the smooth limit arc $\mathcal{L}_k$ that connects neighboring points $A_k = w(l_k)$ and $A_{k+1} = w(l_{k+1})$.

Let us briefly describe two ways to write the equation of the limit arc $\mathcal{L}_k$ that is the image of the vertex $a_k = 0$ of the polygon under mapping (1) when z is approaching the vertex in various directions.

1. Denote $D_\alpha = D_\alpha(\infty) = \{z : |\arg z| < \alpha\pi/2\}$, $0 < \alpha < 2$. If $0 < \alpha < 1$, then $D_\alpha(1)$ is understood as the interior of the convex polygon that is a part of $D_\alpha(\infty)$, and

$$D_\alpha(\rho) = \rho D_\alpha(1), \ \ \rho \geq 1, \ \text{ and } \ D_\alpha(\infty) = \lim_{\rho\to\infty} D_\alpha(\rho).$$

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