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Generalized Solutions and Generalized Eigenfunctions of Boundary-Value Problems on a Geometric Graph

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Abstract—We consider generalized solutions to boundary-value problems for elliptic equations on an arbitrary geometric graph and their corresponding eigenfunctions. We construct analogs of Sobolev spaces that are dense in L_2. We obtain conditions for the Fredholm solvability of boundary-value problems of various types, describe their spectral properties and conditions for the expansion in generalized eigenfunctions. The results presented here are fundamental in studying boundary control problems of oscillations of multiplex jointed structures consisting of strings or rods, as well as in studying the cell metabolism.

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The consideration of generalized solutions for analyzing boundary-value problems defined on stratifiable sets ([1], P. 74; [2], P. 157) allows one to better understand the physical nature of oscillatory processes. This fact has lead to the dissemination of similar ideas in studying boundary-value problems on geometric graphs (networks). Below with the help of integral identities we consider generalized solutions to boundary-value problems for differential equations on an arbitrary geometric graph Γ and the corresponding eigenfunctions from the class $W_2^1(\Gamma)$ (here $W_2^1(\Gamma)$ is the space of functions that belong to $L_2(\Gamma)$ and have generalized derivatives in $L_2(\Gamma)$). This approach implies the construction of analogs of Sobolev spaces which are dense sets in $L_2(\Gamma)$. Analogously, in the classical papers by O. A. Ladyzhenskaya and V.I. Smirnov ([3], P. 38; [4], P. 61; [5], P. 322; [6], P. 438) one considers generalized solutions to boundary-value problems of various types. For these problems we obtain an analog of the energy inequality (see the terminology in [4], P. 91; [6], P. 444) and use it for proving uniqueness theorems. We also establish conditions for the Fredholm solvability of boundary-value problems. In some cases, which are important for applications, we describe the structure of sets of eigenvalues and generalized eigenfunctions. The obtained results are basic in studying problems of the boundary control of oscillatory processes in complex network-like systems associated with a geometric graph; they continue the study commenced in [7, 8].

1. GENERALIZED SOLUTIONS OF EQUATIONS ON GRAPHS

1.1. Preliminary assertions. Here and below we use notions and denotations given in monographs [9] (P. 117), [10] (P. 24). We consider an arbitrary graph, possibly, containing cycles.

Assume that Γ is a connected compact graph-tree, ξ_0 is its root, $V = \{\xi_0, \xi_1, \dots, \xi_m\}$ is its vertex set, and $\Re = \{\gamma_1, \gamma_2, \dots, \gamma_m\}$ is its edge set, the length of each edge being equal to one. A vertex is said to be boundary, if it belongs to only one edge (a boundary edge), all the rest vertices and edges are said to be interior; we denote the set of boundary vertices by $\partial\Gamma$ and do the set of interior vertices by $J(\Gamma)$; $V = \partial\Gamma \cup J(\Gamma)$. We define a partial ordering on Γ in a natural way, namely, for two points a_1 and a_2 the correlation $a_1 \le a_2$ is defined, if a_1 lies on a unique path connecting the root ξ_0 with a_2. Denote $[\omega, \varpi] = \{z \in \Gamma : \omega \le z \le \varpi\}$; if $[\omega, \varpi]$ is an edge γ, then ω is the initial point of γ and ϖ is the

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