Russian Mathematics (Iz. VUZ) *Izvestiya VUZ. Matematika*
Vol. 44, No. 9, pp.1–6, 2000 *UDC 517.12.252*

NON-COMPACTNESS MEASURES IN INVESTIGATION OF INEQUALITIES

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Introduction

In [1] (p. 74) the Banach spaces E_0, E, and E_1 were considered, besides, $E_0 \subset E \subset E_1$, and the embedding of E_0 into E is compact. Then, for every $\varepsilon > 0$, a constant c_ε can be found such that with all $v \in E_0$ one has

$$\|v\|_E \le \varepsilon \|v\|_{E_0} + c_\varepsilon \|v\|_{E_1}. \tag{1}$$

In [2] (p. 154) a result by Yu.A. Dubinskiĭ was exposed, where E_0 was replaced with the set $\Im$ equipped with the function $M : \Im \to R^+$; besides, $\Im \subset E \subset E_1$, $M(v) \ge 0$,

$$M(\lambda v) = |\lambda| M(v) \tag{2}$$

for all $v \in \Im$, the set $\Im_1$ is relatively compact in E, where

$$\Im_k = \{v \in \Im : M(v) \le k\} \tag{3}$$

for all $k \ge 0$. In these conditions for each $\varepsilon > 0$ a constant c_ε can be found such that

$$\|u - v\|_E \le \varepsilon (M(u) + M(v)) + c_\varepsilon \|u - v\|_{E_1} \tag{4}$$

for all u, v from $\Im$.

Remark 1 (see [2], p. 154). Let $\Im = E_0$, M be the norm in E_0. Then inequality (4) yields inequality (1).

We introduce the quantities

$$\tau(U) = \sup_{v \in U} M(v), \quad \widetilde{\tau}(U) = \inf_{v \in U} M(v).$$

Definition 1. A set $U \subset \Im$ is said to be M-bounded if $\tau(U) < \infty$.

Definition 2. We shall say that a set $U \subset \Im$ possesses the M-property if it is M-bounded and $\widetilde{\tau}(U) > 0$.

Definition 3. An operator (in general, nonlinear) $A : \Im \to E$ is said to be M-bounded if any M-bounded set $U \subset \Im$ is sent by the operator A into a bounded subset of the space E.

Without assuming (2) and also that the embedding of $\Im_k$ into E is compact we shall investigate the conditions for the validity of the assertion: For each $\varepsilon > 0$ a constant c_ε can be found such that

$$\|Au - Av\|_E \le \varepsilon (M(u) - M(v)) + c_\varepsilon ||u - v\|_{E_1} \tag{5}$$

for all u, v from an arbitrary set $U \subset \Im$, possessing M-property.