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On Representations of the Weil–Deligne Group

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Abstract—We study admissible orthogonal and symplectic representations of the Weil–Deligne group $\mathcal{W}'(\overline{K}/K)$ of a local non-Archimedean field K. As an application of the obtained results we show that the root number of the tensor product of two admissible symplectic representations of $\mathcal{W}'(\overline{K}/K)$ is 1.

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1. INTRODUCTION

In this paper, we study admissible orthogonal and symplectic representations of the Weil–Deligne group $\mathcal{W}'(\overline{K}/K)$ of a non-Archimedean local field K. The basis of our investigation is the fact that each admissible indecomposable representation of the Weil–Deligne group has a unique irreducible subrepresentation. From the definition of admissible representations of the group $\mathcal{W}'(\overline{K}/K)$ it follows immediately that each admissible representation of the group $\mathcal{W}'(\overline{K}/K)$ is a direct sum of admissible indecomposable subrepresentations. In turn, it is known that each admissible indecomposable representation of the group $\mathcal{W}'(\overline{K}/K)$ is of the form $\alpha \otimes \mathrm{sp}(n)$, where α is an irreducible representation of the Weil group $\mathcal{W}(\overline{K}/K)$ of the field K, n is a positive integer, and the representation $\mathrm{sp}(n)$ is given by formula (3.2) on p. 49. One can easily show that $\alpha \otimes \mathrm{sp}(n)$ has a unique irreducible subrepresentation. In other words, the socle of the representation $\alpha \otimes \mathrm{sp}(n)$ is irreducible. Therefore, it is natural to begin the study of admissible orthogonal and symplectic representations of the group $\mathcal{W}'(\overline{K}/K)$ with the study of orthogonal and symplectic representations (of a group G over a field k) which can be written as direct sums of indecomposable subrepresentations with irreducible socles (see Theorem 2.1). As a result, we have obtained the following theorem.

Theorem 1.1. *If σ' is an admissible minimal symplectic or orthogonal representation of the group $\mathcal{W}'(\overline{K}/K)$, then either*

$$\sigma' \cong (\beta \otimes \mathrm{sp}(m)) \oplus \left(\beta^* \otimes \omega^{1-m} \otimes \mathrm{sp}(m)\right) \tag{1.1}$$

for some positive integer m and irreducible representation β of the group $\mathcal{W}(\overline{K}/K)$, or

$$\sigma' \cong \alpha \otimes \mathrm{sp}(n) \tag{1.2}$$

for some positive integer n and irreducible representation α of the group $\mathcal{W}(\overline{K}/K)$ such that the representation

$$\alpha \otimes \omega^{\frac{n-1}{2}} \begin{cases} \textit{is symplectic} & \textit{if } \sigma' \textit{ is symplectic and } n \textit{ is odd,} \\ \textit{orthogonal} & \textit{if } \sigma' \textit{ is symplectic and } n \textit{ is even,} \\ \textit{orthogonal} & \textit{if } \sigma' \textit{ is orthogonal and } n \textit{ is odd,} \\ \textit{is symplectic} & \textit{if } \sigma' \textit{ is orthogonal and } n \textit{ is even.} \end{cases} \tag{1.3}$$

Here a *minimal* symplectic (or orthogonal) representation is a representation which cannot be written in the form of an orthogonal sum of its nonzero invariant subrepresentations, β^* means the representation contragradient to a representation β, and ω is the one-dimensional representation of the group $\mathcal{W}(\overline{K}/K)$ defined by (3.1).

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