Motion of paticles near black holes and high energy collisions

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Killing vectors

Metric is invariant under transformations

$$x^{\alpha} \to \tilde{x}^{\alpha} = x^{\alpha} + \xi^{\alpha}\delta\lambda \qquad \tilde{g}_{\alpha\beta}(x^{\gamma}) = g_{\alpha\beta}(x^{\gamma})$$

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0$$

$$u^{\alpha}$$

$$\frac{dI}{d\tau} = 0 \qquad I = u_{\alpha}\xi^{\alpha}$$

four-velocity tangent to geodesics

$$\frac{dI}{d\tau} = (u_{\alpha}\xi^{\alpha})_{;\beta}u^{\beta} = \xi^{\alpha}u_{\alpha;\beta}u^{\beta} + u^{\alpha}u^{\beta}\xi_{(\alpha;\beta)} = 0$$

$$ds^2 = -N^2dt^2 + g_\phi(d\phi - \omega dt)^2 + \frac{dr^2}{A} + g_\theta d\theta^2,$$

Metric does not depend on t, ϕ

Two integrals of motion $E = -p_\mu \xi^{(t)\mu}$

Angular momentum $L = p_\mu \xi^{(\phi)\mu}$

Killing vectors responsible for time translations and rotation

$$\xi^{(t)\mu} = (1, 0, 0, 0) \qquad \xi^{(\phi)\mu} = (0, 1, 0, 0)$$

Event horizon

Hypersurface $f(t, x^1, x^2, x^3) = 0$

$df = \partial_\mu f dx^\mu = 0$ dx^μ taken within this hypersurface

Normal vector $n_\mu \sim \partial_\mu f$

Lightlike hypersurface $n_\mu n^\mu = 0$

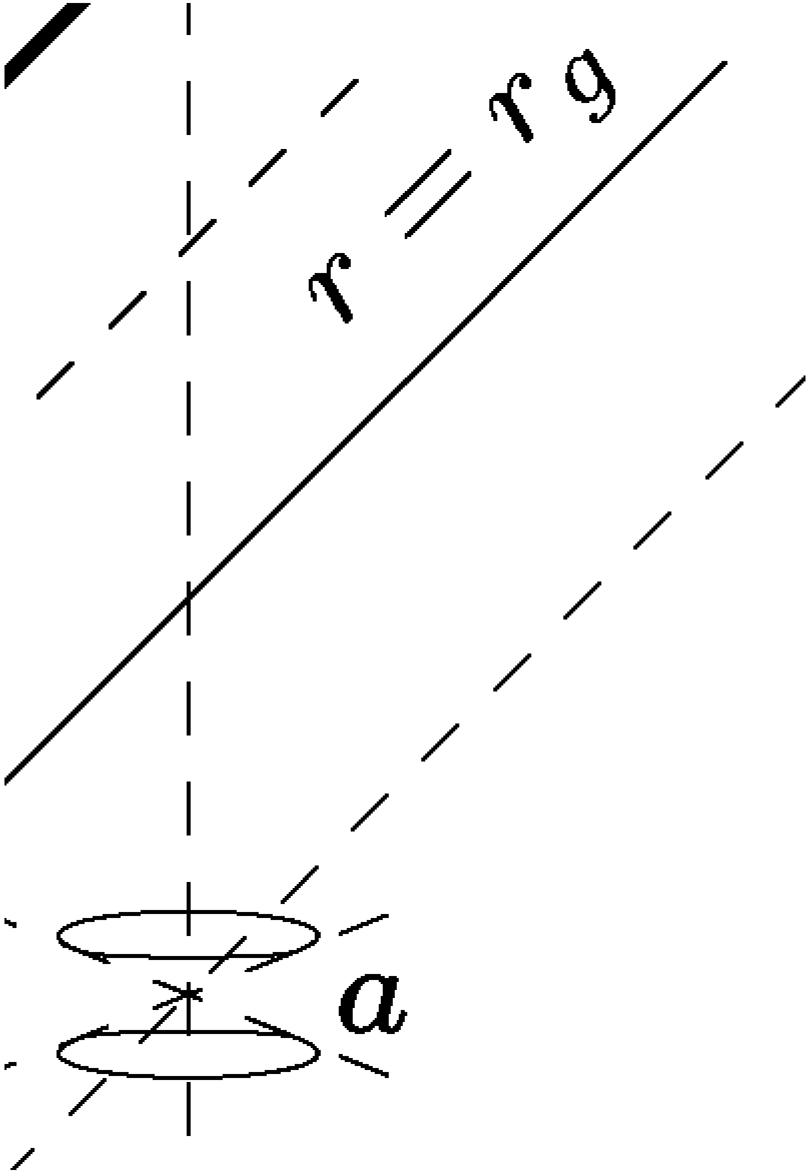
$n_\mu \sim dx^\mu$ In this direction, $ds^2 = dx^\mu dx_\mu = 0$

tangent to leg of light cone

One-way membrane

Trivial case: $t = \pm x$

Nontrivial: closed, r=const



Metric

$$g_{00} = -N^2 + g_\phi \omega^2 \qquad g_{0\phi} = -\omega g_\phi$$

Contravariant components

$$g^{00} = -\frac{1}{N^2} \qquad g^{0\phi} = -\frac{\omega}{N^2}$$

$$g^{\phi\phi} = \frac{1}{g_\phi} - \frac{\omega^2}{N^2}$$

Equations of motion

$$u^t = g^{00} u_t + g^{0\phi} u_\phi$$

$$m u^t = \frac{X}{N^2} \qquad m u^\phi = \frac{L}{g_\phi} + \frac{\omega X}{N^2}$$

$$\frac{d\phi}{dt} = \omega + \frac{L X}{g_\phi} N^2 \qquad X = E - \omega L$$

Observer on circular orbit

$$u^\phi, u^t \qquad \theta = const = \frac{\pi}{2}$$

$$\frac{d\phi}{dt} = \omega + \frac{LX}{g_\phi} N^2$$

Zero angular observer (ZAMO)

L=0 $$\frac{d\phi}{dt} = \omega$$ Locally nonrotating frame (LNRF)

$$ds^2 = -N^2 dt^2 + g_\phi (d\phi - \omega dt)^2 + \frac{dr^2}{A} + g_\theta d\theta^2,$$

Dragging effect

L=0 but $\frac{d\phi}{dt} \neq \omega$

If $\frac{d\phi}{dt} = 0$ $L \neq 0$

More on kinematics in curved space-time

Let a particle (observer) move with the four-velocity U^μ

Locally, it defines the hypersurface orthogonal to it

Induced metric $h_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu$ $\quad h_{\mu\nu}U^\nu = 0$ $\quad h_{\mu\nu}h^{\nu\sigma} = h^\nu_\mu$

$d\tau_{obs} = -dx^\mu U_\mu.$ If $d\tau_{obs} = 0$ two events are simultaneous

proper distance $dl^2 = h_{\mu\nu}dx^\mu dx^\nu$

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dl^2 - d\tau^2_{obs}$$

Local velocity $V^2 = \left(\frac{dl}{d\tau_{obs.}}\right)^2.$ $\quad d\tau^2 = d\tau^2_{obs}(1 - V^2).$

Examples of frames.

Comoving observer

Observer who is sitting at rest with respect to a given coordinate frame:

$$x^i = const \qquad i = 0,1,2,3$$

$$U^\mu = (U^0,0,0,0). \qquad g_{00}(U^0)^2 = -1,$$

$$U^\mu = \frac{1}{\sqrt{-g_{00}}}(1,0,0,0). \qquad U_0 = -\sqrt{-g_{00}}\,,\; U_i = \frac{g_{i0}}{\sqrt{-g_{00}}}. \qquad h_{0i} = 0,$$

$$h_{00} = 0,$$

$$h_{ik} = g_{ik} - \frac{g_{0i}g_{0k}}{g_{00}}$$

condition of simultaneity

$$dt = -dx^i \frac{g_{0i}}{g_{00}}$$

Landau and Lifshitz eq. 84.7 eq. (84.14)

ZAMO frame (zero angular momentum observer)

and its properties

By definition $U_i = 0$ If i is angular variable, standard ZAMO

Normalization condition gives us

$$U^\mu = \frac{1}{N}(1, N^i, 0, 0)$$

$$V^i = \frac{U^i}{U^0} = N^i = \left(\frac{dx^i}{dt}\right)_{osb}.$$

$$U_\mu = -N(1, 0, 0, 0).$$

$$d\tau_{obs} = -dx^\mu U_\mu = N dt.$$

From $U_i = \frac{g_{i0}}{N} + \frac{g_{ik}N^k}{N} = 0$ we find

$$g_{0i} = -g_{ik}N^k \qquad U_0 = -N = \frac{g_{00}}{N} + \frac{g_{0i}N^i}{N}$$

$$g_{00} = -N^2 + g_{0i}N^i N^k.$$

Result:

$$ds^2 = -dt^2N^2 + g_{ij}(dx^i - N^i dt)(dx^j - N^j dt)$$

$$h_{00} = g_{00} + N^2, h_{0i} = g_{0i}, h_{ik} = g_{ik}.$$

$$ds^2 = -dt^2N^2 + g_{\phi\phi}(d\phi - \omega dt)^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2.$$

$$g_{0\phi} = -g_{\phi\phi}\omega, \qquad h_{00} = \omega^2 g_{\phi\phi}, h_{0\phi} = g_{0\phi}, h_{\phi\phi} = g_{\phi\phi}$$

$$N^\phi = \omega, N^r = N^\theta = 0.$$

world line of the ZAMO is orthogonal to the hypersurface t=const

Proof $n_\mu \sim t_{,\mu}$ $n_i = 0$ Thus vectors n_μ and U_μ

Within this hypersurface, $n_\mu dx^\mu = n_i dx^i = 0.$

Energy and momentum in different frames

$$E = -m u_\mu \xi^\mu = -m u_0$$ Killing energy, observer at infinity

Energy, measured by local observer with four-velocity U_μ

$$E_{rel.} = -m u_\mu U^\mu \equiv m\gamma = -m(u_0 U^0 + u_i U^i),$$

whence $$E - m u_i V^i = \frac{E_{rel.}}{U_0} \qquad V^i = \frac{U^i}{U^0}$$

Energy in terms of relative velocity

$$\gamma = -u_\mu U^\mu = \frac{U^\mu dx_\mu}{d\tau} = \frac{d\tau_{obs}}{d\tau} \qquad d\tau^2 = d\tau_{obs}^2(1 - V^2).$$

$$\gamma = \frac{1}{\sqrt{1-w^2}}, E_{rel.} = \frac{m}{\sqrt{1-w^2}}.$$

$$E - p_i V^i = \frac{m}{\sqrt{1-w^2}} \frac{1}{U_0}, \qquad p^\mu = m u^\mu$$

Flat spacetime

$$U^0 = \frac{1}{\sqrt{1-V^2}} = \gamma$$

$$E_{(0)} = \frac{m}{\sqrt{1-w^2}} = \left(E - \vec{p}\vec{V}\right)\gamma,$$

Static observer

$$E = E_{rel.}\sqrt{-g_{00}} = \frac{m\sqrt{-g_{00}}}{\sqrt{1-w^2}}$$

eq. (88.9) of Landau and Lifshiitz

ZAMO observer

$$E - m u_i V^i = E_{rel.} N = \frac{mN}{\sqrt{1-w^2}}$$

In the axially symmetric case, the angular momentum of a particle

is conserved $m u_\phi = L$

$$E - \omega L = E_{rel.} N = \frac{mN}{\sqrt{1-w^2}}.$$

Ergoregion $g_{00} > 0$

$$ds^2 = -N^2 dt^2 + g_\phi (d\phi - \omega dt)^2 + \frac{dr^2}{A} + g_\theta d\theta^2,$$

Is it possible for particle to have $\phi = const$?

$$ds^2 = g_{00} dt^2 + \frac{dr^2}{A} + g_\theta d\theta^2,$$

In ergoregion, all terms positive Interval spacelike, impossible

Outside ergoregion one can choose static frame.

$g_{00} < 0$

$$E = E_{rel.} \sqrt{-g_{00}} = \frac{m\sqrt{-g_{00}}}{\sqrt{1-w^2}} > 0$$

But inside, $g_{00} > 0$ this formula does not work

Energy can be positive or negative

Boundary of ergoregion $g_{00} = 0$

Limit of staticity, infinite redshift

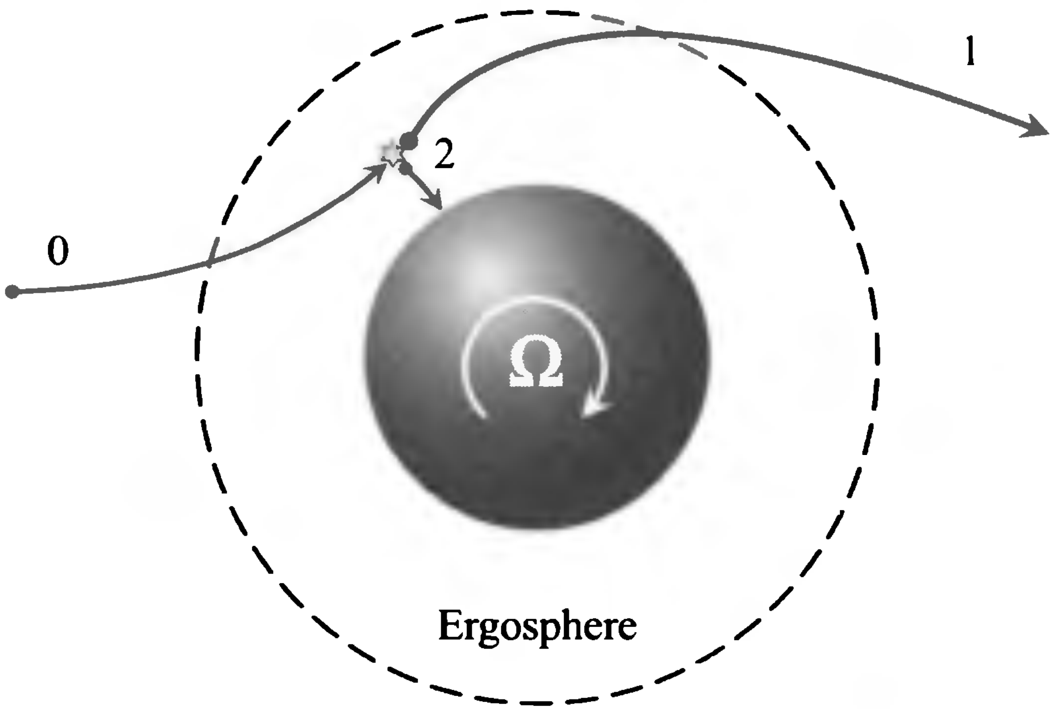
$$\nu_0 = \nu\sqrt{-g_{00}}\,.$$

For a static observer

in the limit $g_{00} \to 0$ and any finite ν

$$\nu_0 \to 0$$

But for moving observer this is not so



Penrose process. A particle 0 enters the ergosphere and decays there into two particles, 1 and 2.

One of them with a negative energy (2) falls into the black hole. The other one (1) escapes the ergosphere with an energy exceeding an energy of the original particle.

Penrose process $E_0 = E_1 + E_2 \qquad E_2 < 0, E_1 > E_0$

Circular orbits

$$u^\mu = \frac{\eta^\mu}{\sqrt{-\eta_\alpha \eta^\alpha}} \qquad u^t u^\phi \neq 0 \qquad \frac{d\phi}{dt} = \frac{u^\phi}{u^t} = \Omega$$

$$\eta^\mu = \xi_{(t)}{}^\mu + \Omega \xi_{(\phi)}{}^\mu$$

Surface at which $u^\mu u_\mu = 0$

$$\xi_{(t)}{}^\mu \xi_{(t)\mu} + 2\Omega \xi_{(\phi)}{}^\mu \xi_{(t)\mu} + \Omega^2 \xi_{(\phi)}{}^\mu \xi_{(\phi)\mu} = 0$$

$$\xi_{(t)}{}^\mu \xi_{(t)\mu} = g_{\mu\nu} \xi_{(t)}{}^\mu \xi^\mu{}_{(t)} = g_{00} \qquad \xi_{(t)}{}^\phi \xi_{(t)\phi} = g_{\mu\nu} \xi_{(\phi)}{}^\mu \xi^\mu{}_{(\phi)} = g_\phi$$

$$\xi_{(t)}{}^\mu \xi_{(\phi)\mu} = g_{\mu\nu} \xi_{(t)}{}^\mu \xi^\mu{}_{(\phi)} = g_{0\phi}$$

$$g_{00} + 2\Omega g_{0\phi} + \Omega^2 g_{\phi\phi} = 0$$

$$\Omega_{\pm} = -\frac{g_{0\phi}}{g_{\phi}} \pm \frac{\sqrt{g^{2}{}_{\phi} - g_{00} g_{\phi}}}{g_{\phi}}$$

$$g^{2}_{\phi} - g_{00} g_{\phi} = g_{\phi} N^{2}$$

$$\Omega_{\pm} = \omega \pm \frac{N}{\sqrt{g_{\phi}}}$$

$$u^{\mu} u_{\mu} \sim (\Omega - \Omega_{+})(\Omega - \Omega_{-})$$

Time-like $u^{\mu} u_{\mu} < 0$ if

$$\Omega_{-} < \Omega < \Omega_{+}$$

$$\Omega_{\pm} = \omega \pm \frac{N}{\sqrt{g_{\phi}}}$$

Outside ergoregion, $g_{00} = -N^2 + \omega^2 g_{\phi} < 0$

$$\Omega_{-} < 0, \Omega_{+} > 0$$

Ω can have any sign

Corotate or counterrotate

Inside ergoregion

$$\Omega_{-} > 0, \Omega_{+} > 0$$

$\Omega > 0$ Only corotates with black hole

Nonequatorial motion, motion in r and θ directions

$$u^{\mu} \sim \eta^{\mu} + \alpha h_{(r)}{}^{\mu} + \beta h_{(\theta)}{}^{\mu}$$

$$g_{00} + \alpha^2 + \beta^2 + 2\Omega g_{0\phi} + \Omega^2 g_{\phi\phi} = 0$$

$$\Omega_{\pm} = -\frac{g_{0\phi}}{g_{\phi}} \pm \frac{\sqrt{g^2{}_{\phi} - \tilde{g}_{00} g_{\phi}}}{g_{\phi}}$$

$$\tilde{g}_{00} = g_{00} + \alpha^2 + \beta^2 \qquad \Omega_{-} > 0$$

corotates

Kerr metric

$$ds^2 = -dt^2(1 - \frac{r_g r}{\rho^2}) + (r^2 + a^2 + \frac{r_g r a^2}{\rho^2}\sin^2\theta)\sin^2 d\theta^2 +$$

$$\frac{\rho^2 dr^2}{\Delta} + \rho^2 d\theta^2 - \frac{2 r_g r}{\rho^2} a \sin^2\theta d\phi dt$$

$$r_g = 2M \qquad \Delta^2 = r^2 - r r_g + a^2 \qquad \rho^2 = r^2 + a^2\cos^2\theta$$

At large distances $-g_{00} \approx 1 - \frac{2M}{r}$

$$-g_{0\phi} \approx \frac{r_g a}{r}\sin^2\theta \approx \frac{2J}{r}\sin^2\theta \qquad J = Ma$$

$$r_+ = M + \sqrt{M^2 - a^2}$$

$a < M$ Nonextremal black holes

$$r_- = M - \sqrt{M^2 - a^2}$$

$a = M$ Extremal black holes

$$r_+ = r_-$$

$a > M$ No black hole, naked singularity

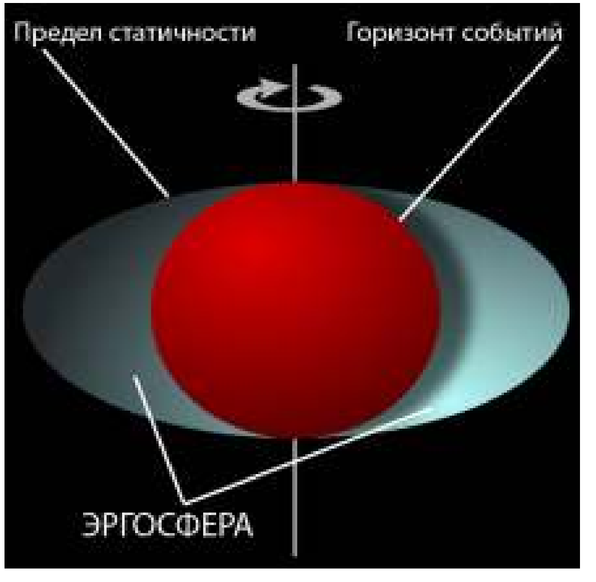
$a = 0$ Schwarzschild metric $r_+ = r_g = 2M$

$$ds^2 = -dt^2\left(1 - \frac{r_g}{r}\right) + \frac{dr^2}{1 - \frac{r_g}{r}} + r^2 \sin^2\theta d\phi^2 + r^2 d\theta^2$$

$$-g_{00} = 1 - \frac{r_g r}{\rho^2}$$

Ergosphere: $g_{00} = 0$

$$r r_g = \rho^2$$



Effective potential

$$m u^t = \frac{X}{N^2} \qquad m u^\phi = \frac{L}{g_\phi} + \frac{\omega X}{N^2}$$

$$X = E - \omega L$$

Equatorial motion

$$\theta = const = \frac{\pi}{2}$$

$$g_{\mu\nu} u^\mu u^\nu = -1$$

$$g_{rr} \left(\frac{dr}{d\tau} \right)^2 + \ldots = -1$$

$$\left(\frac{dr}{d\tau}\right)^2 + V_{eff} = 0$$

$$-V_{eff} = \varepsilon^2 + \frac{2M}{r^3}(a\varepsilon - L)^2 + \frac{a^2\varepsilon^2 - L^2}{r^2} - \frac{\Delta}{r^2}\delta$$

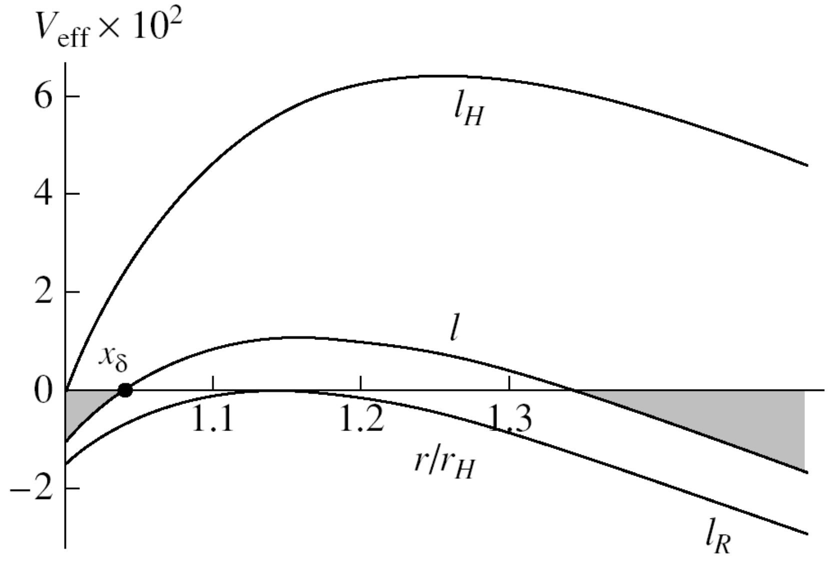
$E = m\varepsilon, J = mL$ $\delta = 1$ for time-like geodesics

$\delta = 0$ for light-like geodesics

If particle falls from infinity with E=m I can reach horizon provided

$$l_L \le l \le l_R \qquad l = \frac{L}{M} \qquad A = \frac{a}{M} \le 1$$

$$l_L = -2(1 + \sqrt{1 + A}) \qquad l_R = 2(1 + \sqrt{1 - A})$$



The effective potential for $A = 0.95$ and $l_R \approx 2.45$, $l = 2.5$, $l_H \approx 2.76$. Allowed zones for $l = 2.5$ are shown by the gray color.

High energy processes near BHs

Key quantity: energy in centre of mass frame

1 particle $$m^2 = \left| P_\mu P^\mu \right|$$

2 particles colliding in some point

$$E^2{}_{cm} = \left| P_\mu P^\mu \right|$$

Total momentum $$P_\mu = p^{(1)}{}_\mu + p^{(2)}{}_\mu$$

$$P_a = (E_{c.m.}, 0, 0, 0) \qquad u^\mu u_\mu = -1$$

Individual E finite, energy in CM frame unbound

Two different kinds of energy

Killing energy $E = -p_\mu \xi^\mu$ ξ^μ Killing vector

E conserved, integral of motion since metric is static or stationary

Energy in the CM frame $E_{c.m.}$

not conserved. Moreover, it is defined in one point only.
point of collision

Collision of two particles: general formulas

$$E_{c.m.}^2 = -P^\mu P_\mu \qquad P^\mu = p_1^\mu + p_2^\mu$$

$$E_{c.m.}^2 = m_1^2 + m_2^2 + 2m_1 E_{rel.}(2,1) = m_1^2 + m_2^2 + 2m_2 E_{rel.}(1,2),$$

$$E_{rel.}(1,2) = m_1\gamma \qquad E_{rel.}(2,1) = m_2\gamma \qquad \gamma = -u_1^\mu u_{2\mu},$$

Local three-velocity

In special relativity $u^\mu = \frac{dx^\mu}{ds} = (1, v^i)\gamma,\ \gamma = \frac{1}{\sqrt{1-v^2}}.$

$$v^i = u^i\sqrt{1-v^2} \qquad v^i = \frac{dx^i}{d\tau_{obs.}}, \qquad d\tau^2 = d\tau_{obs}^2(1-v^2).$$

Curved space-time

tetrad basis $h^{\mu}_{(a)}$ Let $U^{\mu}=h^{\mu}_{(0)}$ and $h^{\mu}_{(i)}$ be orthogonal to it

Special relativity

$$v^i = \frac{dx^i}{d\tau_{obs.}},$$

Natural definition

$$v^{(i)} = \frac{h^{(i)}_{\mu} dx^{\mu}}{-U^{\mu} dx_{\mu}},$$

$$v^{(i)} = \frac{u^{\mu} h^{(i)}_{\mu}}{-u^{\mu} h_{(0)\mu}}.$$

Massless case

No comoving frame, v=c=1, m=0 $E = \hbar \nu$

Photon with wave four-vector k_μ

$\nu_0 = -k_\mu \xi^\mu$ is conserved along the trajectory

$\nu_0 - k_i V^i = \frac{\nu}{U^0}$, $\nu = -k_\mu U^\mu$ velocity of observer U^μ

for static observer $V^i = 0$ $U^0 = \frac{1}{\sqrt{-g_{00}}}$ $\nu_0 = \nu\sqrt{-g_{00}}\,.$

For ZAMO observer $V^\phi = \omega$ $k_\phi = L$

$U^0 = \frac{1}{N}$ $\nu_0 - \omega L = \nu N.$

ν has the meaning of frequency.

Types of collisions

Head-on collisions   Motion towards black hole (BSW)   Intermediate case: circular orbits

Head-on collision

1975 - 1977   T. Piran, J. Katz and J. Shanam

Two particles move in opposite directions near BH

Almost infinite relative blue shift

E in CM frame almost diverges

Special scenario. Particle near black (not white) hole moving away from horizon and colliding with another particle

BSW effect, its physical explanation and properties

Universal character of BSW effect near BH

Kinematic nature of the BSW effect. Role of critical trajectories

BSW effect and acceleration horizons

Geometric explanation

Kinematic explanation for collisions inside BH

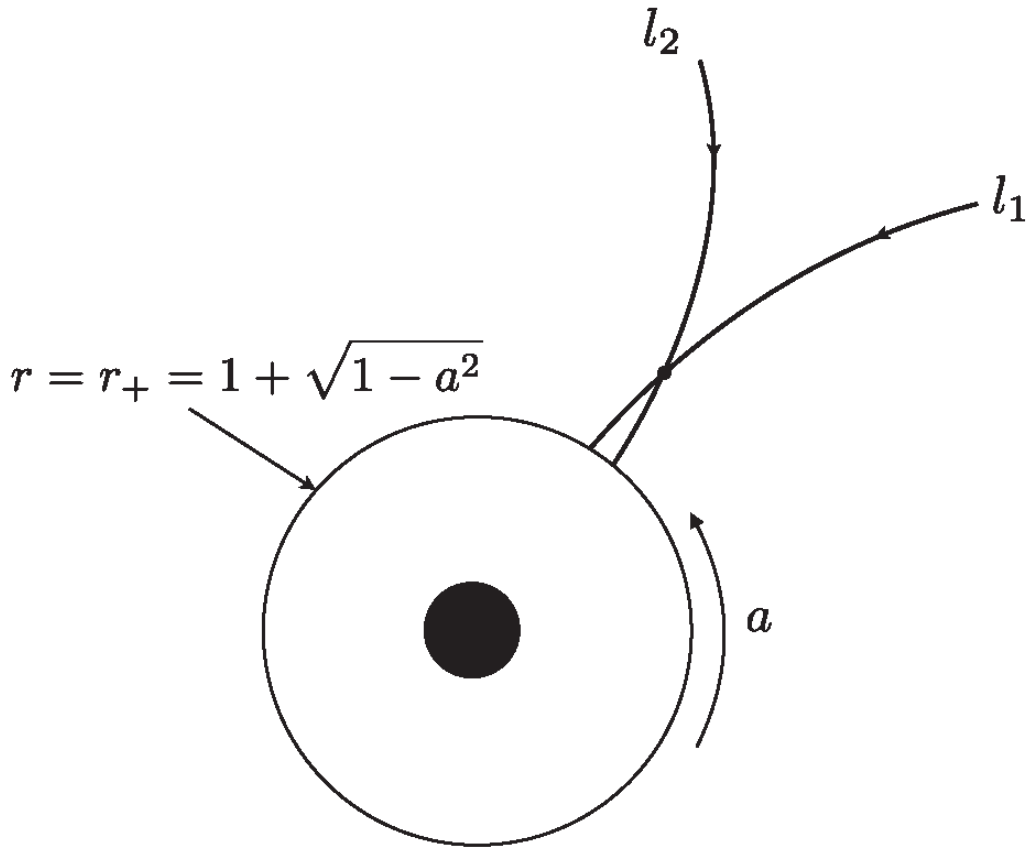
Extremal versus nonextremal BHs

Kinematic censorship
B

BSW effect versus Penrose process: what can be seen at infinity?

Role of self-force due to gravitational radiation

M. Banados, J. Silk, and S. M. West PRL 2009



$$-2(1+\sqrt{1+a}) < l < 2(1+\sqrt{1-a})$$

Both particles experience blue shift, centre of mass frame is in free fall.

$$
(E_{\mathrm{c.m.}}^{\mathrm{Kerr}})^2 = \frac{2m_0^2}{r(r^2 - 2r + a^2)}[2a^2(1 + r) - 2a(l_2 + l_1) - l_2 l_1(-2 + r) + 2(-1 + r)r^2 - \sqrt{2(a - l_2)^2 - l_2^2 r + 2r^2}\sqrt{2(a - l_1)^2 - l_1^2 r + 2r^2}].
$$

M=1, l=L/mM

Acceleration of particles as universal property of rotating black holes

O. Z., PRD 2010

Role of horizon

Universality of black hole physics

Unified approach to nonextremal versus extremal black holes

Energy in CM frame

$$E_{c.m.}{}^2 = -(m_1 u^\mu{}_1 + m_2 u^\mu{}_2)(m_1 u_{1\mu} + m_2 u_{2\mu})$$

$$E_{c.m.}{}^2 = m^2{}_1 + m_2{}^2 + 2 m_1 m_2 \gamma$$

$$\gamma = -(u_1 u_2)$$

$$ds^2 = -N^2 dt^2 + g_\phi (d\phi - \omega dt)^2 + \frac{dr^2}{A} + g_\theta d\theta^2,$$

equatorial plane $\theta = \frac{\pi}{2} \quad (z = 0)$ Is a symmetry one

$u_0 = -E \qquad u_\phi = L$ conserved quantities

Integrals of geodesic equations

$$g_{\mu\nu} u^\mu u^\nu = -1$$

$$\dot{t} = u^0 = \frac{E - \omega L}{N^2} = \frac{X}{N^2}. \qquad X = E - \omega L$$

Forward in time condition X>0 or X=0 If N=0 (horizon)

$$\dot{\phi} = \frac{L}{g_{\phi\phi}} + \frac{\omega X}{N^2}, \qquad \dot{r}^2 = \frac{(E - \omega L)^2}{N^2} - 1 - \frac{L^2}{g_{\phi\phi}}.$$ for timelike trajectories

$$\frac{E_{c.m.}^2}{2m^2} = \frac{X_1 X_2 - \varepsilon_1 \varepsilon_2 Z_1 Z_2}{N^2} + 1 - Y, \qquad Z_i = \sqrt{(E_i - \omega L_i)^2 - N^2 b_i}, \, b_i = 1 + \frac{L_i^2}{g_{\phi\phi}},$$

$$Y = \frac{L_1 L_2}{g_{\phi\phi}}.$$

$\varepsilon = -1$ for particle moving towards horizon

$\varepsilon = +1$ away from horizon

$$\varepsilon_1\varepsilon_2 = -1$$

head-on collision, Piran et al

$$\frac{E^2_{c.m.}}{2m^2} = \frac{X_1X_2 + Z_1Z_2}{N^2} + 1 - Y,$$

$E^2_{c.m.}$ always unbound near horizon

For any relationship between energies and angular momenta

$$\varepsilon_1 = \varepsilon_2 = -1$$

BSW

$$\frac{E_{c.m.}^2}{2m^2} = \frac{X_1 X_2 - Z_1 Z_2}{N^2} + 1 - Y,$$

$$X_i \equiv E_i - \omega L_i$$

$$Z_i = \sqrt{(E_i - \omega L_i)^2 - N^2 b_i}, \; b_i = 1 + \frac{L_i^2}{g_{\phi\phi}}, \qquad Y = \frac{L_1 L_2}{g_{\phi\phi}}.$$

In general case, $E_{c.m.}^2$ remains bound in horizon limit $N \to 0$

Special conditions for unbound $E_{c.m.}^2$

Two kinds of particles (trajectories)

Usual $X_H \equiv E - \omega_H L \neq 0$

Critical $X_H \equiv E - \omega_H L = 0$

Different limiting transitions

1) Let, for generic L_i one approaches the horizon, so $N \to 0$

$$\left(\frac{E_{c.m.}^2}{2m^2}\right)_H = 1 + \frac{b_{1(H)}(L_{2(H)} - L_2)}{2(L_{1H} - L_1)} + \frac{b_{2(H)}(L_{(1)H} - L_1)}{2(L_{2(H)} - L_2)} - \frac{L_1 L_2}{(g_{\phi\phi})_H}, L_{i(H)} \equiv \frac{E_i}{\omega_H}.$$

$$L_1 = L_{1(H)}(1-\varepsilon)$$

$$\left(\frac{E_{c.m.}^2}{2m^2}\right)_H \approx \frac{b_{1(H)}(L_{2(H)} - L_2)}{2L_{1(H)})\varepsilon}.$$

$$\lim_{L_1 \to L_{1(H)}} \lim_{N \to 0} E_{cm} = \infty.$$

2) Let us take $L_1 \rightarrow L_{1(H)}$ first and, then, consider the limit $N \rightarrow 0$

$$\frac{E_{c.m.}^2}{2m^2} \approx \frac{(E_2 - \omega_H L_2)}{N}[B_1 \frac{E_1}{\omega_H} - \sqrt{(\frac{E_1^2}{\omega_H^2} B_1^2 - b_1)}].$$

$$\lim_{N \to 0} \lim_{L_1 \to L_{1(H)}} E_{cm} = \infty.$$

1 usual particle and 1 (near) critical

Proper time to approach horizon:

$$\tau \sim \int \frac{dl N}{Z} \sim l \rightarrow \infty$$

Extremal versus nonextremal

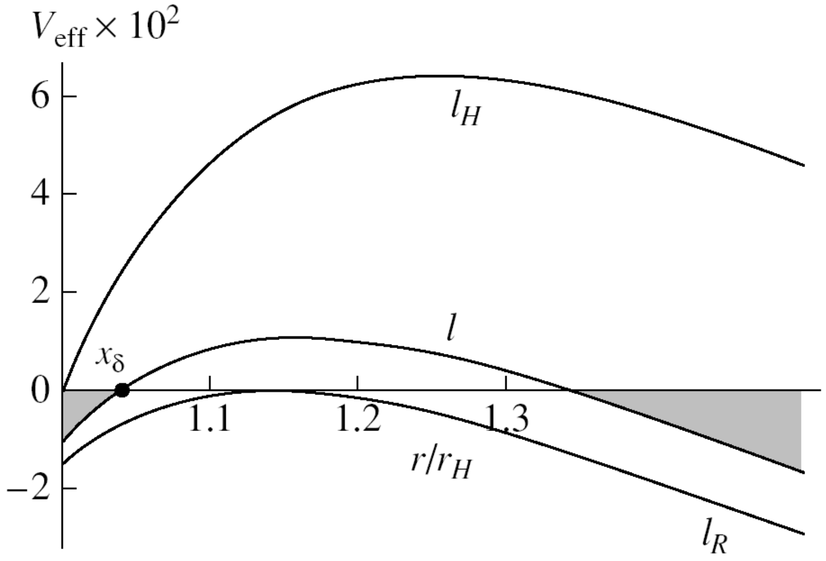
Problems with attaining extremality, a=0,998 (Thorne)

Jacobson et al, Berti at al: difficulties in realization

Grib and Pavlov: nonextremal Kerr

Extremal case: collision near horizon

$$E_{c.m.} \approx \frac{m}{\sqrt{\delta}} \sqrt{\frac{2(L_H - L_2)}{1 - \sqrt{1 - a^2}}} \qquad L_1 = L_{(H)} - \delta$$



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Multiple scattering (Grib and Pavlov)

$$-2\left(1+\sqrt{1+a}\,\right)=L_L \le L \le L_R = 2\left(1+\sqrt{1-a}\,\right). \qquad a = 1-\varepsilon$$

$$L_H - L_R = 2\,\frac{\sqrt{1-a}}{a}\left(\sqrt{1+a}+\sqrt{1-a}-a\right) \approx 2(\sqrt{2}-1)\sqrt{\varepsilon}$$

$$\varepsilon \to 0$$

Some estimates

Collision of two protons

(Grib and Pavlov, Piattela)

Extremal black hole

Extremal BH of solar mass, To obtain in CM energy of Grand Unification

10^{24} sec > time of Universe 10^{18} sec

To obtain 10^{3} larger than at LHC 10^{8} sec

Nonextremal black hole

a=0.998 BH 10^{8} solar mass

To obtain in CM energy of Grand Unification 10^{6} sec

13 days

Head on collisions (Piran et l) no
fine tuning

BSW effect – requires fine tuning

Two issues: getting unbound energy in CM frame, measurements at infinity

Two key ingredients of BSW effect (O..Z., 2011-2012)

1) Horizon

2) Special class of trajectories (critical particles)

Collisions between 1 critical and 1 usual particle produce BSW effect

General properties of BSW effect for nonextremal black holes

$$ds^2 = -N^2 dt^2 + g_{\phi\phi}(d\phi - \omega dt)^2 + dl^2 + g_{zz} dz^2.$$

$$\dot{t} = u^0 = \frac{E - \omega L}{N^2}. \qquad \dot{\phi} = \frac{L}{g_{\phi\phi}} + \frac{\omega(E - \omega L)}{N^2},$$

$$\frac{E_{c.m.}^2}{2m^2} = h + 1 - \frac{L_1 L_2}{g_{\phi\phi}}$$

$$h = \frac{X_1 X_2 - Z_1 Z_2}{N^2}, \qquad E_1 = \omega_+ L_1 (1 + \delta)$$

$$\frac{E_{c.m.}^2}{2m^2} \approx (X_2)_+ \frac{b_+}{\omega_+ L_+ \delta} f(\chi). \qquad f = \frac{1}{2\cos^2 \frac{\eta}{2}}.$$

$N \sim \sin \eta$

On horizon f=1/2

Turning point f=1

Circular orbits and BSW effect

$$V_{eff}(\rho_0) = 0, \qquad \frac{dV_{eff}}{d\rho}(\rho_0) = 0$$

Nonexistence of near-horizon circular orbits for generic nonextremal rotating black holes

$$-\frac{1}{2}\frac{dV_{eff}}{d\rho} = [-(E-\omega L)L\frac{d\omega}{d\rho} - \frac{dN}{dl}b - \frac{N^2}{2}\frac{db}{d\rho}].$$

$$b = 1 + \frac{L^2}{g_{\phi\phi}}$$

Horizon limit

$N \to 0$ $\qquad \frac{dN}{dl} \to \kappa.$ $\qquad$ Surface gravity, constant

$$-\frac{1}{2}\frac{dV_{eff}}{d\rho} \to -b_+\kappa \neq 0$$

$h = \frac{X_1 X_2}{N^2}.$

Innermost stable circular orbit (ISCO)

$$\frac{d^2 V_{eff}(\rho_0)}{d\rho^2} = 0.$$

$$x \approx H\kappa^{2/3}, H = const.$$

$$x = \rho - \rho_+$$

$$X \sim N \sim \kappa^{\frac{2}{3}}.$$

O-variant of BSW effect: collisions on circular orbits

Particle 1 on circular orbit $h = \frac{X_1 X_2}{N^2}.$ $E_{c.m.} \sim\sim \kappa^{-1/3}.$

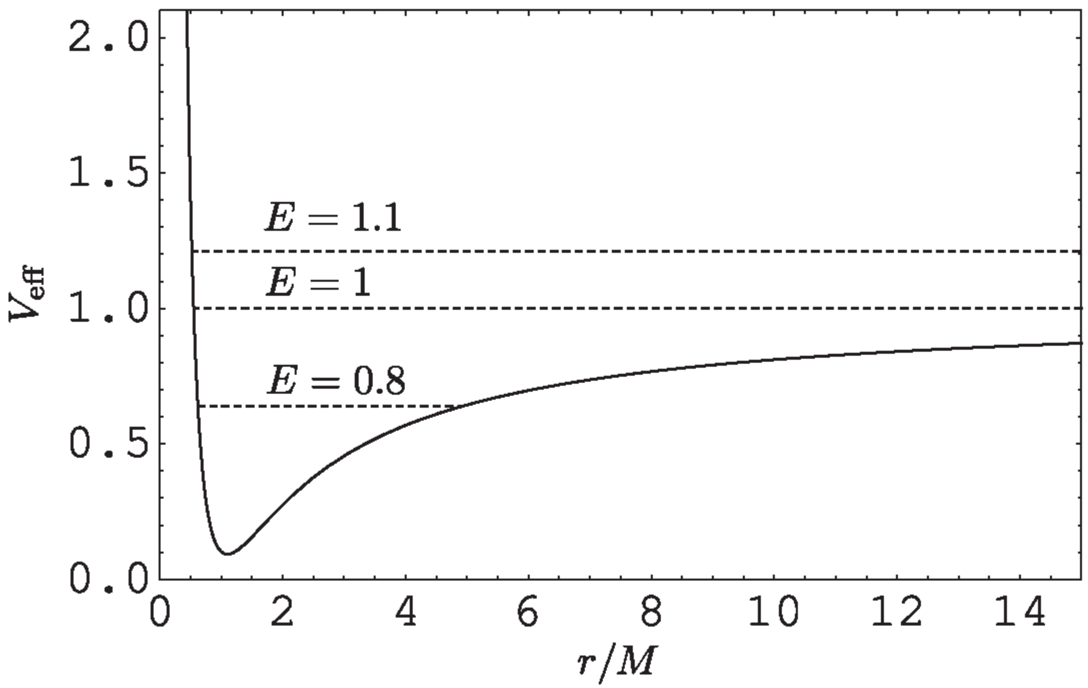
H-variant of BSW effect: collisions of particles plunging from circular orbits

$$E_{c.m.} \sim \kappa^{-1/2}.$$

Another mechanisms

Patil, Joshi, Kimura, Nakao

RN metric, naked singularity



$$Q \approx M$$

$$Q > M$$

Black hole

$$Q < M$$

Naked singularity

$$E_{c.m.}^2 = \frac{4m^2}{1 - \frac{M^2}{Q^2}}.$$

Small N

Small f in point of collision

No horizons, no singularities (Patil, Joshi)

Extension to rotating case O. Z.

$$m_1 m_2 \gamma = \frac{X_1 X_2 + \delta Z_1 Z_2}{N^2} - \frac{L_1 L_2}{g_\phi} - g_\theta p_1^\theta p_2^\theta.$$

$\delta = -1$ motion in same direction

$\delta = +1$ motion in opposite directions, head-on

Small N, large energy in CM frame

$$ds^2 = -N^2 dt^2 + g_\phi (d\phi - \omega dt)^2 + \frac{dr^2}{A} + g_\theta d\theta^2,$$

Both particles usual, proper time bounded

Role of gravitational radiation

Naively: it bounds the growth of E in CM, restricts BSW effect

More careful inspection: under rather general assumptions (radial acceleration is finite in OZAMO frame, asimuthal force tends to zero not too slowly) the critical trajectories do exist. As a consequence, the BSW effect persists.

Details: I. V. Tanatarov and O. Z., **PRD 2013**

BSW effect survives!

BSW effect versus Penrose process

Particles 1 and 2 move towards BH, collide and produce particles 3 and 4

What energy can be observed at infinity?

Large $E_{c.m.}$ but strong redshift In static case $\omega\sqrt{-g_{00}} = \omega_0$

M. Bejger et al, Harada et al (Kerr spacetime), O. Z. (dirty BH)

Rotaing extremal black holes

Conservation laws (energy and radial momentum)

Upper limits for specific reactions

Elastic collisions $m_1 = ... m_4 = m$

$\lambda_+ = \max E_3 = 1.343m$ $\lambda_+ = 1.343m$

$$\eta = \frac{E_3}{E_1 + E_2} \leq 0.67$$

If 1=3, 2=4, we have a new free parameter

Maximal extraction = 1.176

Extraction $$\eta = \frac{E_3}{E_1 + E_2}.$$

Is it possible to achieve this inequality?

In the Kerr case

Scenario IN+

$$\eta_m = \frac{2(2+\sqrt{3})}{q+2} \approx 1.466$$

In two other scenarios no energy extraction

Universal upper bound

Collisions near inner horizon

Two particles collide inside black hole

$$ds^2 = -dt^2 f + \frac{dr^2}{f} + r^2 d\omega^2.$$

RN $$f = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} = (1 - \frac{r_+}{r})(1 - \frac{r_-}{r})$$

$$r_- \leq r < r_+ \qquad f = -g \leq 0 \qquad r \equiv -T \qquad t \equiv y$$

$$ds^2 = -\frac{dT^2}{g(T)} + g(T)dy^2 + T^2 d\omega^2.$$

Initial moment $r_- < r \leq r_1 < r_+$ $r \equiv -T$

Later, r decreases

Collisions near $r = r_-$

Formally, one can achieve

$$\lim E_{c.m.}(r) = \infty$$ when $r \to r_-$

However, by itself this does NOT mean that the effect occurs

There is also kinematic condition that collision does occur

Collision

$$T_1 = T_2 \qquad\qquad y_1 = y_2$$

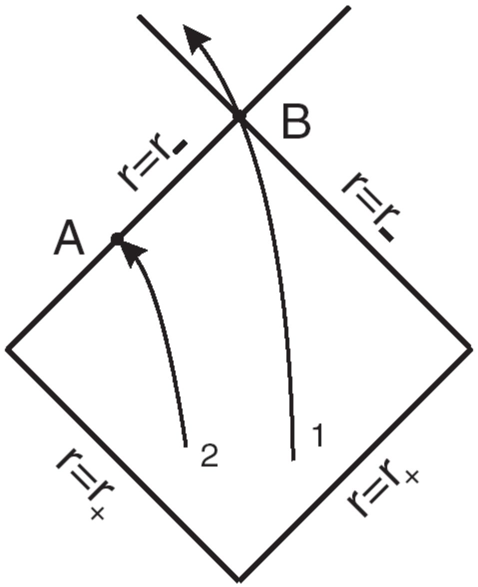
Carter-Penrose diagram, for fixed r different points

$$(U_1, V_1) \qquad\qquad (U_2, V_2)$$

Kruskal-like coordinates, analytic extension

Collisions near inner horizon

Again, one of two particle should be critical. Then, the following cases are possible.



Kinematic censorship preserved

Fig. 1. Impossibility of strong version of BSW effect. Critical particle 1 passes through bifurcation point whereas usual one 2 hits left horizon

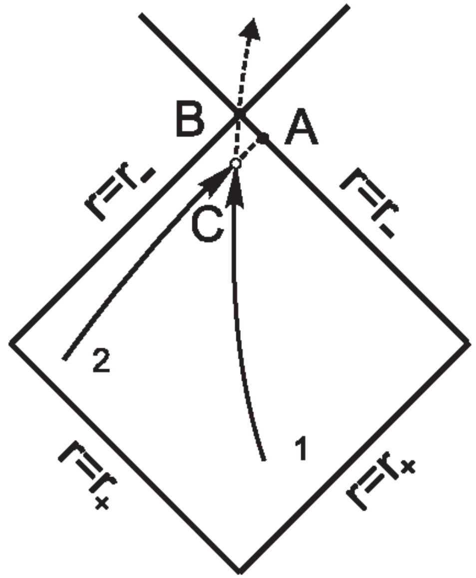
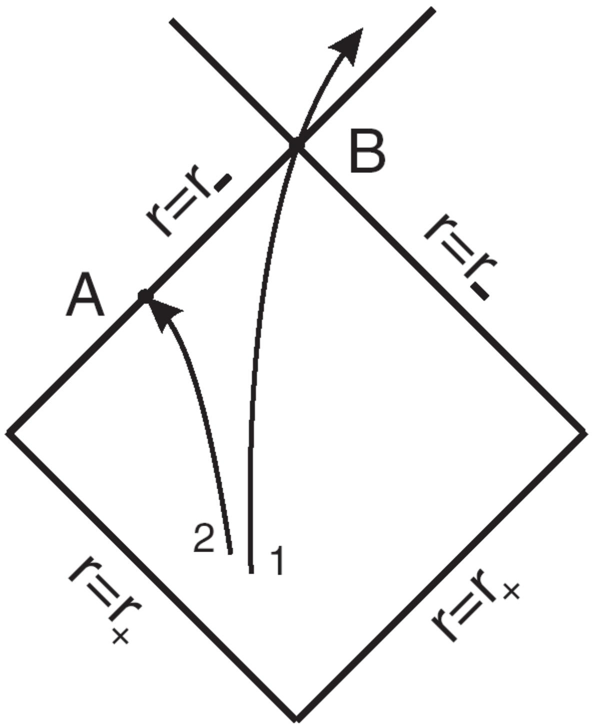


Fig. 2. The weak version of BSW effect. Near-horizon collision between Critical particle 1 and usual one 2.



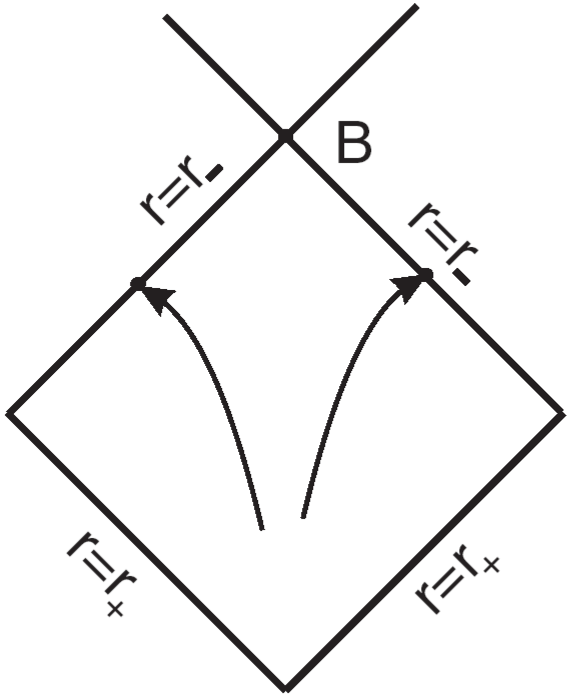
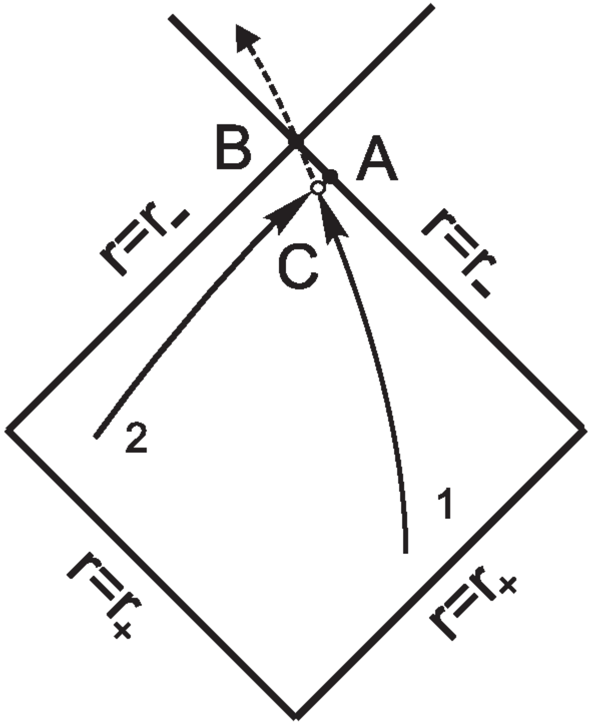


Fig. 4. Impossibility of strong version of PS effect. Two usual particles hit different branches of horizon.

Fig. 3. Impossibility of strong version. Critical particle 1 passes through bifurcation point, whereas
a usual one 2 hits left horizon.

Kinematic censorship



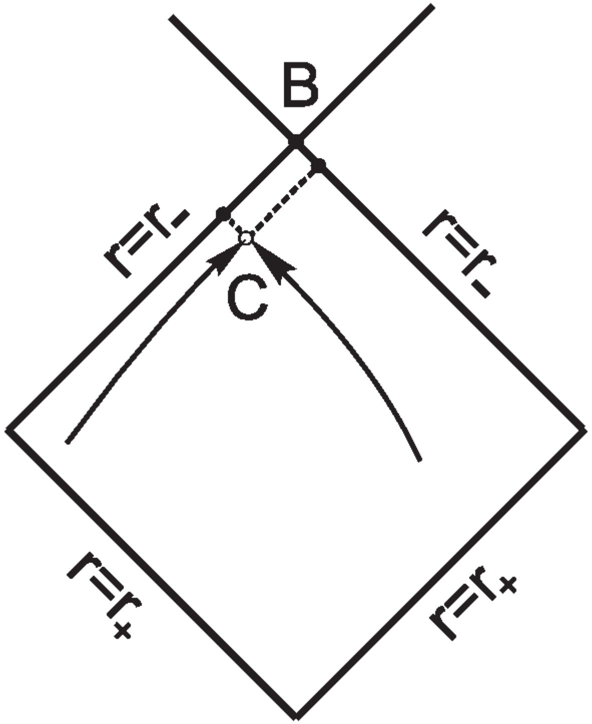


Fig. 5. Weak version of effect. Near-horizon collision between critical particle 1 and usual one 2

Fig. 6. Weak version of effect. Collision between two usual particles near left horizon.

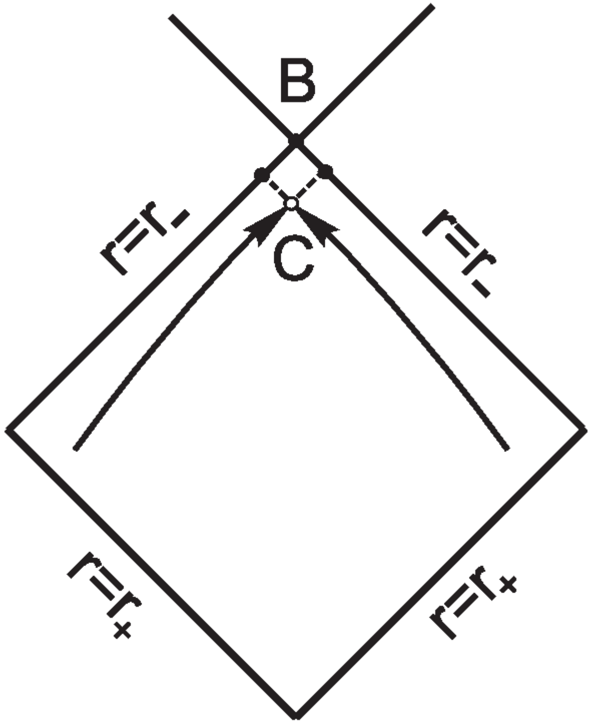


Fig. 7. Weak version of effect. Collision between two usual particles near bifurcation point.

Analog in flat space-time

Minkowski and Milne metrics

$ds^2 = -dt^2 + dx^2.$ $x = \tilde{t}\sinh\tilde{x},$ $t = \tilde{t}\cosh\tilde{x},$

$t < 0, |x| < |t|.$ $ds^2 = -d\tilde{t}^2 + \tilde{t}^2 d\tilde{x}^2.$ $\tilde{t}^2 = t^2 - x^2,$ $\tanh\tilde{x} = \frac{x}{t}.$

right horizon $x = -t$ $\tilde{t} = 0, \tilde{x} = -\infty,$

left one $x = +t$ $\tilde{t} = 0, \tilde{x} = +\infty.$

bifurcation point $x = 0 = t$ $\tilde{t} = 0, |\tilde{x}| < \infty.$

$$X = m u_\mu \xi^\mu \qquad x - Vt = x_0. \qquad \tilde{t} = \frac{X}{\cosh \tilde{x}_0 (V \cosh \tilde{x} - \sinh \tilde{x})} = \frac{x_0}{\sinh \tilde{x} - V \cosh \tilde{x}},$$

$$X = -x_0 E, E = \frac{1}{\sqrt{1 - V^2}}.$$

critical if X=0

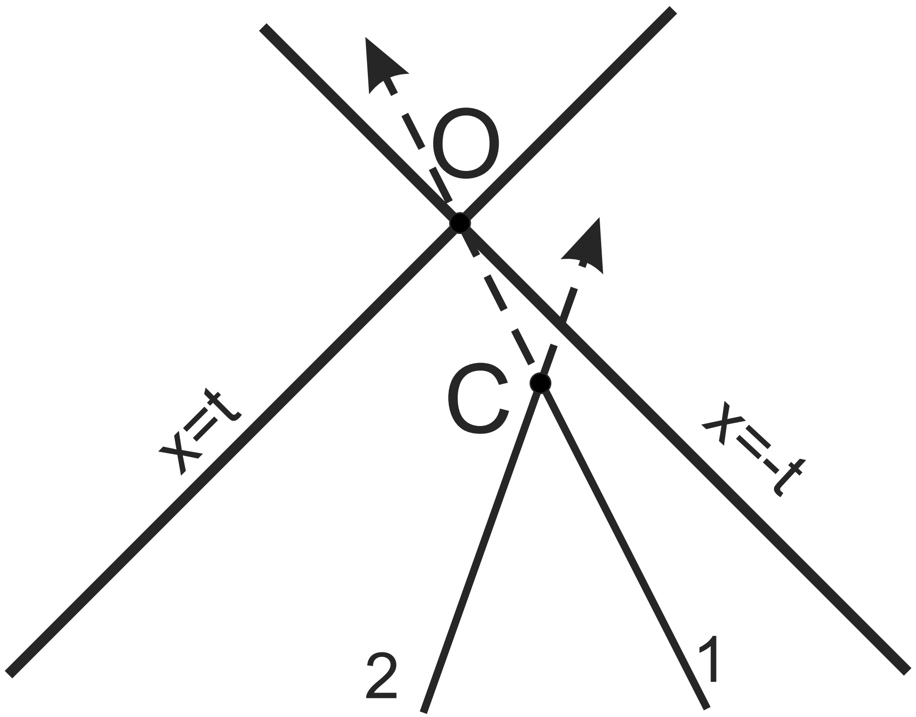
near-critical if X small

usual X=O(1)

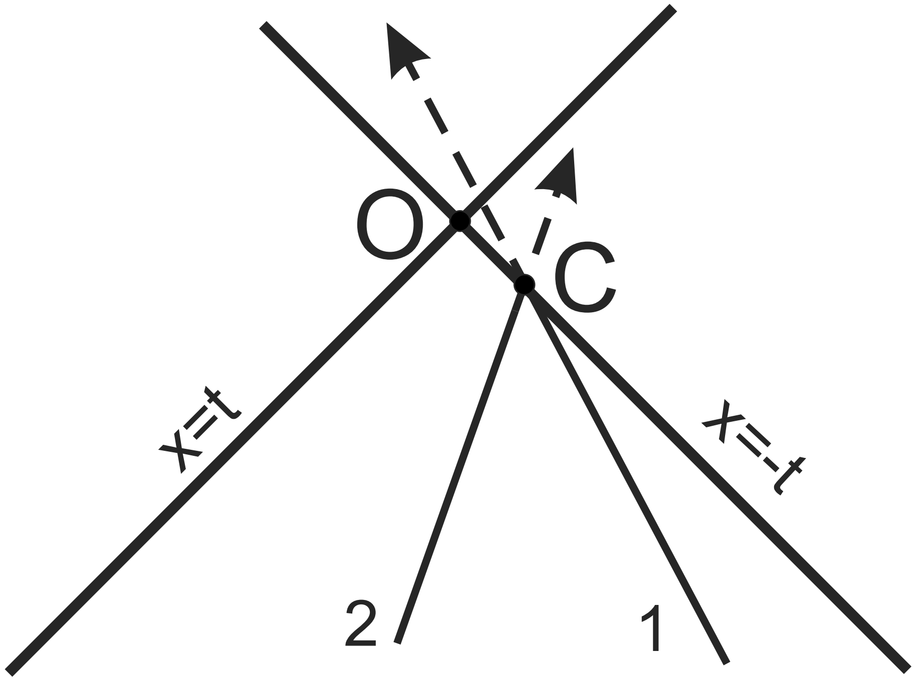
Types of collisions leading to the BSW effect.

Scenario	V_1	V_2	X_1	X_2	β_1	β_2	Location
Aa	$\approx +1$	≈ -1	< 0	> 0	$2\lvert X_1 \rvert$	$\sim \tilde{t}_c^2 \ll \beta_1$	Bifurcation point
Ab	≈ -1	$\approx +1$	> 0	< 0	$\sim \tilde{t}_c^2$	$2\lvert X_2 \rvert \gg \beta_1$	Bifurcation point
Ba	intermediate	≈ -1	0	> 0	$\sim \lvert \tilde{t}_c \rvert$	$\sim \tilde{t}_c^2 \ll \beta_1$	Bifurcation point
Bb	intermediate	$\approx +1$	0	< 0	$\sim \lvert \tilde{t}_c \rvert$	$2\lvert X_2 \rvert \gg \beta_1$	Bifurcation point
Ca	≈ -1	intermediate	0	< 0	$\sim \lvert \tilde{t}_c \rvert$	$2\lvert X_2 \rvert \gg \beta_1$	Horizon
Cb	$\approx +1$	intermediate	0	> 0	$\sim \lvert \tilde{t}_c \rvert$	$\sim \tilde{t}_c^2 \ll \beta_1$	Horizon

Similar effects for cosmological horizons



Collision between the critical particle 1 and a usual particle 2
near the horizon



Collision between the near-critical particle 1 and a usual particle 2 on the horizon

Alternative mechanisms of getting unbound energies in CM frame

Collisions inside ergosphere, not near horizon

Finite Killing energy E, large negative angular momentum L

Grib and Pavlov 2013 (Kerr metric)

O. Z. 2013 (generalization)

$$ds^2 = -N^2 dt^2 + g_{\phi\phi}(d\phi - \omega dt)^2 + \frac{\rho^2}{\Delta} dr^2 + g_{\theta\theta} d\theta^2.$$

$\omega > 0$ Equatorial plane:

$$m_1 m_2 \gamma = \frac{X_1 X_2 - \varepsilon_1 \varepsilon_2 Z_1 Z_2}{N^2} - \frac{L_1 L_2}{g}.$$

$$Z^2 = X^2 - N^2 (m^2 + \frac{L^2}{g}),$$

For BSW – small denominator. Now – large numerator

Large $|L|$ possible for L<0 since $X = E - \omega L \geq 0$

We want E to be fixed

From geodesic equations

$$Z^2 = \frac{L^2 g_{00}}{g} - 2E\omega L + E^2 - N^2 m^2 (1 + g_{\theta\theta}\dot{\theta}^2) \geq 0,$$

We want to have finite E and large negative L

Outside ergosphere this is impossible since $g_{00} < 0$ there

So we must look what happens inside ergosphere (or on boundary)

$$ds^2 = -N^2 dt^2 + g_{\phi\phi}(d\phi - \omega dt)^2 + \frac{\rho^2}{\Delta} dr^2 + g_{\theta\theta} d\theta^2.$$

$$\omega > 0$$

$$m\dot{\phi} = \frac{\omega E}{N^2} - L\frac{g_{00}}{gN^2},$$

Inside ergosphere, $g_{00} > 0$

If $L = -|L| < 0$ and $|L|$ increases

$\frac{d\phi}{d\tau}$ also increases!

Collisions inside ergosphere

$$E_{c.m.}^{2} \approx \frac{2|L_{2}|g_{00}}{N^{2}g}[\varepsilon_{1}\sqrt{(L_{1})_{+}-L_{1}}-\varepsilon_{2}\sqrt{(L_{1})_{-}-L_{1}}]^{2}.$$

$g_{00} > 0$

Arbitrarily large

If $|L|$ so is

Kinematic explanation

$$E - \omega L = \frac{mN}{\sqrt{1 - V^2}},$$

Horizon limit $N \to 0$ $V_{us} \to 1$ $V_{cr} < 1$

Inside ergosphere, large L<0 for particle 1 (critical)

$V_{us} < 1$ $V_{cr} \to 1$

CONCLUSION

High energy collisions due to horizon

Outside black hole
Inside black hole
Role of critical trajectories
ISCO
Force does NOT spoil BSW effect, critical trajectories survive
RN metric: example of significant effect at infinity
Relevant physical factors: BH rotation, electric charge, magnetic field
Universality typical of BH physics

Alternative mechanisms

No horizon but system in some sense “close” to its appearance
Ergoregion

A need for further studies of Penrose process in combination with BSW effect