ISSN 1066-369X, Russian Mathematics (Iz. VUZ), 2014, Vol. 58, No. 7, pp. 61–70. © Allerton Press, Inc., 2014.
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Sourcewise Representability Conditions and Power Estimates of Convergence Rate in Tikhonov's Scheme for Solving Ill-Posed Extremal Problems

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Received December 29, 2012

Abstract—We study the rate of the convergence of approximations generated by the Tikhonov scheme for solving ill-posed optimization problems with smooth functionals given in a general form in a Hilbert space. We establish sourcewise representability conditions which are necessary and sufficient for the convergence of approximations at a power rate. Sufficient conditions are related to the estimate of the discrepancy with respect to the objective functional, while the necessary ones are formulated for the estimate with respect to the argument. We specify certain cases when sufficient and necessary conditions coincide in essence.

DOI: 10.3103/S1066369X1407007X

Keywords: *ill-posed optimization problem, Hilbert space, Tikhonov's scheme, convergence rate, sourcewise representability condition.*

1. In this paper we study the extremum problem

$$\min\{J(x) : x \in H\}, \tag{1}$$

where J is a bounded from below and weakly semicontinuous from below on H functional and H is a real Hilbert space. In what follows, when necessary, we impose additional conditions on the functional J. Symbols $\|\cdot\|$ and $(\cdot,\cdot)$ denote, respectively, the norm and the scalar product in H. We assume that the solution set X^* of problem (1) is nonempty,

$$X^* = \{x \in H : J(x) = J^*\}, \quad J^* = \inf\{J(x) : x \in H\}.$$

It is well-known (e.g., [1], P. 163) that the stated problem is ill-posed in the sense that the minimized sequence of the functional J, generally speaking, does not converge in the norm of H to the solution set X^*. Therefore, in order to obtain a stable process for generating points which are close either to this set or to some fixed element from X^* in the norm of the space H, we need to apply regularization methods. Most of such methods are based on the Tikhonov scheme. A simplest variant of this scheme consists in choosing an element $\xi \in H$ as an a priori bound for the desired solution $x^* \in X^*$ and in choosing a sequence of regularization parameters $\{\alpha_n\}$, $\alpha_n > 0$, $\lim\limits_{n\to\infty} \alpha_n = 0$. Then we state regularized extremum problems

$$\min\{J_\alpha(x) : x \in H\}; \quad J_\alpha(x) = J(x) + \frac{1}{2}\alpha\|x-\xi\|^2, \quad \alpha > 0, \tag{2}$$

and choose the solution x_α to problem (2) with $\alpha = \alpha_n$ as a current approximation to X^*. In view of conditions imposed above on the functional J, the set $X_\alpha = \operatorname{Argmin}\{J_\alpha(x) : x \in H\}$ is nonempty for any $\alpha > 0$ ([1], theorem 1.3.2), which means that the scheme under consideration is realizable. Denote

$$X^{**} = \operatorname{Argmin}\{\|x-\xi\| : x \in X^*\}.$$

The next theorem is well-known in the theory of ill-posed extremum problems (see, e.g., [1], pp. 180–182).

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