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LAPLACE TRANSFORM AND THE NONCOMMUTATIVE VERSION OF THE PALEY–WIENER THEOREM

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1. In [1] the image with respect to Laplace transform of the space $L^2(S)$ was described, where S is the cone in a locally compact Abelian group. In this article a similar result is established for subsemigroups of non-Abelian locally compact groups. In addition, the Laplace transform is considered in some algebras of "growing" functions. As in the classical case, this gives us a wider domain of definition than that of the Fourier transformation of functions on such a semigroup.

Everywhere in what follows G stands for the unimodular locally compact group of type I (see, e. g., [2]) with the Haar measure ν, S means the generating subsemigroup possessing inner points. We denote by S_1^* a set of all nonnegative bounded semicharacters of the semigroup S, i. e., continuous homomorphisms from S into the multiplicative semigroup $[0,1]$, which differ from identical zero. We shall suppose that $S_1^* \neq \{1\}$ and that any semicharacter $\rho \in S_1^*$ is continued up to (necessarily unique) homomorphism $\rho : G \to (0;\infty)$. In [3] a wide class of the Lie semigroups which satisfy these conditions, was described. In particular, the mentioned conditions are fulfilled when G is a resolvable Lie group and S its Lie subsemigroup invariant with respect to interior automorphisms of the group G. Further, $\widehat{G}$, as usual, means the dual space of the group G, μ the Plancherel measure on $\widehat{G}$ (see [2], [4]). We denote by $\widetilde{S}$ a set (of classes) of representations of the semigroup S of the form $\rho\lambda$, where $\rho \in S_1^*$, $\lambda \in \widehat{G}$, and let $\widetilde{S}_0 = \widetilde{S} \setminus \widehat{G}$, $S_0^* = S_1^* \setminus \{1\}$. Set also $S_c = \{x \in G : \rho(x) \leq 1 \ \forall \rho \in S_1^*\}$. Clearly, S_c is a closed subsemigroup of the group G with a nonempty interior and $(S_c)_c = S_c$. The space S_1^* is assumed to be equipped with the topology of pointwise convergence on S_c. We denote by $L^p(S_c)$ the space (of classes) of functions from $L^p(G)$, concentrated on S_c, by $L^1(S_c, \rho_0 d\nu)$ the space of functions f, concentrated on S_c, such that $f\rho_0 \in L^1(G)$, where $\rho_0 \in S_1^*$.

By the convolution of the functions f, g, concentrated on S_c (if it exists) we shall understand the function

$$f * g(x) = \int_G f(y)(L_y g)(x) d\nu(y) = \int_{S_c \cap x S_c^{-1}} f(y)(L_y g)(x) d\nu(y),$$

where $(L_y g)(x) = g(y^{-1}x)$.

2. Proposition 1. *The space $L^1(S_c, \rho_0 d\nu)$ with the convolution considered as the multiplication is a Banach algebra containing $L^1(S_c)$ in the capacity of a proper subspace if $\rho_0 \neq 1$.*

Proof. The first assertion follows directly from the easily verifiable equality

$$(f\rho_0) * (g\rho_0) = (f * g)\rho_0. \tag{1}$$

Let us choose a point $x_0 \in S$ so that $0 < \rho_0(x_0) < 1$. If the group G is discrete, then the indicator 1_A of the set $A = \{x_0^n : n = 1, 2, \dots\}$ belongs to $L^1(S_c, \rho_0 d\nu)$, but does not belong to $L^1(S_c)$ (x_0 is an element of an infinite order), and the last assertion has been proved.