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ON SHAPIRO AND SHIELDS THEOREM ON ZEROS OF ANALYTIC FUNCTIONS OF BOUNDED GROWTH

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1. Introduction

We consider the result by Shapiro and Shields (see [1]), which by their words goes back to Hayman (though their reference to [2] does not seem to be justifiable) and is cited here in the following slightly strengthened formulation.

Theorem. *If an analytic function $f(z)$, $|z| < 1$, has a limitation on its growth*

$$\log^+ |f(z)| = O((1 - |z|)^{-\lambda}), \quad |z| \to 1 - 0, \tag{1}$$

with a constant $\lambda < 1$, then, for any sequence $\{z_n\}$ of zeros of this function, which converges in a non-tangent way to a certain point ζ, $|\zeta| = 1$, the Blaschke condition is fulfilled

$$\sum_n (1 - |z_n|) < +\infty. \tag{2}$$

In the capacity of a terminological comment we should note the following:

a) it is assumed that to every zero of the function $f(z)$, which is included into the sequence $\{z_n\}$, the number of the corresponding summands in (2) is equal to its multiplicity;

b) the convergence of the sequence $\{z_n\}$ to the point ζ, $|\zeta| = 1$, is assumed to be *non-tangent to the unit circle* if $|\zeta - z_n| = O(1 - |z_n|)$, i.e., for $\{z_n\} \to \zeta$, the points z_n remain inside a fixed angle with the vertex ζ, which is inscribed into the unit circle;

c) $\log^+ a = \begin{cases} \log a & \text{if } a > 1; \\ 0 & \text{if } 0 \le a \le 1. \end{cases}$

The proof of the above theorem is given in [1] only under the assumption that $\lambda < 1/2$ with respect to the exponent in (1); as for the case $1/2 < \lambda < 1$, which requires a more complex technique, it was not considered in this paper (in [1] only some general words were devoted to it). In view of the existing gap between the declared statement and its part proved in fact, references to this work (see, e.g., in [3], [4]) as to a *certainly established* fact cannot be considered as completely correct. For the sake of correctness we should say that in [4] the authors, on the basis of a constraint upon function's growth, which is more general than (1), in the capacity of the proof of Theorem suggested some references to results obtained earlier by them in [5] along with their comments about how to this end those results should be modified.

This article intends, first, to give a *direct* and *elementary* — without an application of special branches of the theory of meromorphic functions — *proof of Theorem* and, second, to illustrate on examples that in its formulation both the requirements — that of the *strict* inequality $\lambda < 1$ and of the non-tangent convergence of the sequence $\{z_n\}$ to the unit circle — are essential.