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The Limit Cycles of a Second-Order System of Differential Equations: The Method of Small Forms

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Abstract—In this paper we investigate the existence of limit cycles of a system of the second-order differential equations with a vector parameter.
We propose a method for representing a solution as a sum of forms with respect to the initial value and the parameter; we call this technique the method of small forms. We establish the conditions under which a sufficiently small neighborhood of the equilibrium point contains no limit cycles. We construct a polynomial, whose positive roots of odd multiplicity define the lower bound for the number of cycles, and simple positive roots (other positive roots do not exist) define the number of limit cycles in a sufficiently small neighborhood of the equilibrium point.
We prove theorems, whose conditions guarantee that a positive root of odd multiplicity defines a unique limit cycle, but a positive root of even multiplicity defines exactly two limit cycles. We propose a method for defining the type of the stability of limit cycles.

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The existence of limit cycles is studied in many papers. The most interesting results are obtained in [1–14]. One of the methods for investigating the problem of limit cycles in a sufficiently small neighborhood of the equilibrium state of a system is the method of small parameter. Its application is fraught with significant computational difficulties, especially in the case of a vector parameter.

In this paper we study the question about the existence of limit cycles in a sufficiently small neighborhood of the equilibrium state by the method of small forms; we do not exclude the case of a vector parameter.

Consider a system of equations in the form

$$\dot{x} = -y + \sum_{i+j=k_0}^{n} (a_{ij} + \mu_{ij})x^i y^j, \quad \dot{y} = x + \sum_{i+j=k_0}^{n} (b_{ij} + \lambda_{ij})x^i y^j, \tag{1}$$

where a_{ij} and b_{ij} are constant real values, μ_{ij} and λ_{ij} are parameters, $k_0 \geqslant 2$.

Assume that $\mu_\nu = (\mu_{\nu 0}, \mu_{\nu-11}, \ldots, \mu_{0\nu})$ and $\lambda_\nu = (\lambda_{\nu 0}, \lambda_{\nu-11}, \ldots, \lambda_{0\nu})$ with any $\nu \in \{k_0, k_0 + 1, \ldots, n\}$, $\gamma_\nu = (\mu_\nu, \lambda_\nu)$ is a vector whose first $\nu+1$ coordinates are equal to coordinates of the vector μ_ν, while the rest $\nu+1$ coordinates are coordinates of the vector λ_ν. Define the vector $\gamma = (\gamma_{k_0}, \gamma_{k_0+1}, \ldots, \gamma_n)$.

In polar coordinates system (1) takes the form

$$\dot{\rho} = \sum_{\nu=k_0}^{n} \rho^\nu [\tau_\nu(\varphi) + \tau_\nu^1(\varphi, \gamma_\nu)], \quad \dot{\varphi} = 1 + \sum_{\nu=k_0}^{n} \rho^{\nu-1}[\lambda_\nu(\varphi) + \lambda_\nu^1(\varphi, \gamma_\nu)]. \tag{2}$$

Let $|z| = \max\limits_i\{|z_i|\}$, $\gamma = re$, $|e| = 1$, $r > 0$, $\gamma_\nu = re_\nu$, $D(\delta_0) = \{(\varphi, \rho, r) : \varphi \in [0, 2\pi], \rho \leqslant \delta_0, r \leqslant \delta_0\}$, $\vartheta = (\alpha, r)$, $|\vartheta| = \max\{\alpha, r\}$, $\alpha \geqslant 0$, $V(\delta_0) = \{\vartheta : |\vartheta| \leqslant \delta_0\}$, $T(\delta_0) = \{(\varphi, \vartheta) : \varphi \in [0, 2\pi], \vartheta \in V(\delta_0)\}$, $\delta_0 > 0$, $S = \{e : |e| = 1\}$.

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