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GENERALIZED RECURRENT SYMMETRIC TENSOR FIELDS

M.V. Smol'nikova

In this article we consider generalized recurrent symmetric tensor fields of type $(2,0)$ on the Riemannian manifolds with sectional curvature of fixed sign and on the Euclidean space. We apply our results to the geometry of the generalized recurrent and concircular recurrent Riemannian manifolds (see [1]).

The results of this article were announced in [2].

1. Introduction

Let us consider an n-dimensional Riemannian manifold (M,g), $n \geq 2$, with the Levi-Civita connection ∇. In [3] K. Yano introduced the torse-forming vector field, i.e., a vector field $\xi \in C^\infty TM$ satisfying the equation $\nabla\xi = \lambda g + \eta \otimes \xi$, where $\lambda \in C^\infty M$, $\eta \in C^\infty T^*M$. Just as the torse-forming vector field is a generalization of the recurrent vector field ($\nabla\xi = \eta \otimes \xi$), the *generalized recurrent symmetric tensor field* $\varphi \in C^\infty S^2 M$, i.e., a field φ satisfying the equation

$$\nabla\varphi = \lambda \otimes g + \eta \otimes \varphi, \tag{1.1}$$

where $\lambda, \eta \in C^\infty T^*M$, is a generalization of the recurrent tensor field ($\nabla\varphi = \eta \otimes \varphi$).

If on (M,g) a recurrent tensor field $\varphi \in C^\infty S^2M$ exists, then on (M,g) a generalized recurrent tensor field $\psi \in C^\infty S^2M$ exists; in fact, we can take the tensor field $\psi = \varphi - \frac{1}{n}(\operatorname{trace}_g \varphi)g$. In particular, if the tensor Ric is recurrent, then the Einstein tensor $G = \operatorname{Ric} - \frac{1}{n}Skg$, where $Sk = \operatorname{trace}_g \operatorname{Ric}$, is a generalized recurrent tensor.

2. Generalized recurrent tensor fields on a manifold with sectional curvature of fixed sign

Let $x^1, \dots, x^n$ be a local coordinate system on (M,g), $\varphi_{ij} = \varphi(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ be the coordinates of a generalized recurrent symmetric tensor field φ. Using the operator of covariant derivative ∇_i with respect to the vector field $\frac{\partial}{\partial x^i}$, we construct a vector field X with coordinates $X^i = (\nabla_k\varphi^{il})\varphi_l^k - (\nabla_l\varphi^{lk})\varphi_k^i$.

With respect to the coordinates $x^1, \dots, x^n$ equation (1.1) has the form $\nabla_k\varphi = \lambda_k g_{ij} + \eta_k\varphi_{ij}$, from this we easily obtain that $\operatorname{div} X = \nabla_i X^i \equiv 0$.

On the other hand, using the Ricci identity

$$\nabla_k \nabla_l \varphi_{ij} - \nabla_l \nabla_k \varphi_{ij} = -\varphi_{pj}R^p_{ikl} - \varphi_{ip}R^p_{jkl},$$

where R^m_{ikl} are the coordinates of the curvature tensor R of ∇, we obtain that

$$\operatorname{div} X = R_{ij}\varphi^{ik}\varphi^j_k - R_{ijkl}\varphi^{ik}\varphi^{jl},$$

where $R_{ij} = R^m_{imj}$ are the Ricci tensor coordinates.