Russian Mathematics (Iz. VUZ) *Izvestiya VUZ. Matematika*
Vol. 49, No. 11, pp.5–13, 2005 *UDC 517.968:519.642*

POLYSINGULAR INTEGRAL EQUATIONS WITH POSITIVE OPERATORS

B.G. Gabdulkhaev

Introduction

It is well known (see, e. g., [1]–[5]) that the theory of multidimensional singular integral equations (SIEs) with *multiple* integrals is extremely difficult and many unsolved problems still exist in this area. For example, the theorems of existence and uniqueness of solution are intricate and often difficult for verification; approximate solution methods are not sufficiently elaborated. In particular, even many well-known approximate methods are not yet theoretically substantiated (except for those rare ones [3]–[8] which have the specific character). The problem of optimization of computational methods [9], [10] for such equations is still open.

In this paper, we study exact and approximate solution methods for multidimensional SIEs with m-multiple ($m \in \mathbb{N}$) Hilbert integrals understood in sense of the Cauchy–Lebesgue principal value. We obtain rather simple and effective sufficient conditions of existence and uniqueness of solutions for such SIEs, and on this base we justify several approximate methods. In particular, we propose the *optimal by order* projective method and investigate its approximative properties.

Note that results of this paper were partly announced in [8] (Chap. 4, § 5).

1. The theorem of existence and uniqueness of solution

Let $C = C(T)$ and $L_2 = L_2(T)$ be the spaces of continuous and, respectively, quadratically summable in the domain $T = [0, 2\pi]^m \subset \mathbb{R}^m$ ($m \in \mathbb{N}$), 2π-periodic in each of m-variables real functions with the usual norms $\|\cdot\|_{C(T)} = \|\cdot\|_\infty$ and $\|\cdot\|_{L_2(T)} = \|\cdot\|_2 \equiv \|\cdot\|$ (where $\|1\| = 1$), and with the scalar product $(\varphi;\psi)$ of elements $\varphi, \psi \in L_2$.

In the space L_2, we consider the polysingular integral equation

$$A\varphi \equiv a(s)\varphi(s) + \frac{1}{(2\pi)^m}\int_T h(s,\sigma)\varphi(\sigma)\operatorname{ctg}\frac{\sigma - s}{2}d\sigma = f(s), \tag{1}$$

where $a(s) \in C(T)$, $f(s) \in L_2(T)$, $h(s,\sigma) \in C(T^2)$ and $\varphi \in L_2(T)$ are, respectively, the given and desired real functions, and $s = (s_1, \dots, s_m)$ and $\sigma = (\sigma_1, \dots, \sigma_m) \in T \subset \mathbb{R}^m$. In addition,

$$\operatorname{ctg}\frac{\sigma - s}{2}d\sigma = \operatorname{ctg}\frac{\sigma_1 - s_1}{2}\cdot \dots \cdot \operatorname{ctg}\frac{\sigma_m - s_m}{2}d\sigma_1 \cdot \dots \cdot d\sigma_m,$$
$$B\varphi \equiv J_m h\varphi \equiv \frac{1}{(2\pi)^m}\int_T h(s,\sigma)\varphi(\sigma)\operatorname{ctg}\frac{\sigma - s}{2}d\sigma, \quad m \in \mathbb{N}.$$

In what follows, we assume that the function $h(s,\sigma) \in C(T^2)$ is such that the generated by it singular integral operator $B : L_2 \to L_2$ is bounded, namely,

$$\|B\| = \|J_m h\| \le M_0 = \text{const} < \infty. \tag{2}$$