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# SOME PROPERTIES OF LINEAR CONTINUOUS TRANSFORMATIONS OF HILBERT SPACES

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*Introduction.* We will denote by $H$ a complex separable Hilbert space with at most countable orthonormed basis $G = \{g_k^0\}_{\mathcal{K}}$. The base of this article is the following

**Proposition 1.** *Any transformation $V \in \mathcal{L}(H)$ defines uniquely the sequence $\{y_k\}_{\mathcal{K}}$ of vectors in $H$ such that*

$$Vx = \sum_{\mathcal{K}} (x, y_k) g_k^0 \quad \forall x \in H; \tag{1}$$

*in addition, $y_k = V^* g_k^0$.*

Indeed, since $Vx \in H$, we have $Vx = \sum_{\mathcal{K}} (Vx, g_k^0) g_k^0 = \sum_{\mathcal{K}} (x, V^* g_k^0) g_k^0$, and it suffices to put $y_k = V^* g_k^0$. The uniqueness of decomposition (1) is obvious.

In the finite-dimensional case decomposition (1) can be found in [1] (p. 72), for the countable-dimensional case the same source gives us a special form of decomposition (1) for compact operators, which is related to a specific choice of the basis $G$ (ibid., p. 202). In the case where $V : X \to H$ (here $X$ is a Banach space and $H = L^2$), decomposition (1) was given in [2] and [3] (theorem 3, p. 11); the case $X = H = L^2$ was noted in [3] (theorem 4, p. 12). Obviously, representation (1) is a generalization of the Riesz–Fisher theorem on representation of linear functionals to the case of linear operators. Moreover, on one hand, to deduce representation (1) we do not need to apply the Riesz–Fisher theorem; on the other hand, (1) immediately yields a new way for deduction of the Riesz–Fisher theorem. Indeed, if $\Phi(x)$ is a linear functional continuous on $H$, then $Vx = \Phi(x) g_1^0$ is the one-dimensional operator continuous on $H$; the comparison of its decomposition (1) and $Vx = \Phi(x) g_1^0$ gives us $\Phi(x) = (x, y_1)$. The uniqueness of $y_1$ follows from the uniqueness of representation (1). However, this is namely the Riesz–Fisher theorem. Note that the proof exposed above can be generalized also to the nonseparable case.

Since the vectors $y_k$, $k \in \mathcal{K}$, determine uniquely $V$, they can be taken as coordinates of $V$. We write $V = (y_k)_{\mathcal{K}}$. The objective of this article is to obtain some new properties of the transformations $V \in \mathcal{L}(H)$ by means of their coordinates.

**I.** What are the conditions under which an arbitrarily given sequence $\{y_k\}_{\mathcal{K}} \subset H$ defines $V \in \mathcal{L}(H)$, $V = (y_k)_{\mathcal{K}}$? A necessary and sufficient condition for this fact in the case $H = L^2$ is given in [2], and [3] (see theorem 4 there). In an abstract formulation, it has the form

$$\sum_{\mathcal{K}} |(x^0, y_k)|^2 < \infty \quad \forall x^0 \in H. \tag{2}$$

Here and in what follows we denote by $x^0, g^0, \dots$ the unit vectors in $H$. Thus, just only the convergence of series (2) ensures that the generated operator $V$ is bounded.