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On One Transformation of Parameter-Spaces of Real Interpolation Method

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Abstract—In connection with estimation of interpolation orbits and coorbits we introduce a new transformation acting in the class of all parameter-space of real interpolation K-method. We "calculate"the result of the transformation of classical parameters. It is revealed that the transformation of weighted L_1-spaces leads to Orlicz spaces.

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1. Denote by R_+ the interval $(0,\infty)$ with the Lebesgue measure. Let D_1 be a cone of all non-increasing integrable (and, therefore, non-negative) functions on R_+. For $v \in D_1$ we write $\|v\|_{D_1}$ or $\|v\|_1$ instead of $\|v\|_{L_1(R_+)}$.

In this paper we use the basic concepts and facts of the theory of interpolation of linear operators ([1, 2] and especially [3]).

Let E be a parameter of real K-method of interpolation, i.e., Banach ideal space (BIS) of measurable functions on R_+, containing "minimal" positive concave function $\min(1,t)$, $t > 0$. If $w = w(t)$ is positive measurable function on R_+, then by E^w we denote weight space E with the weight w of the BIS functions $x = x(t)$ such that $w \cdot x \in E$, $\|x\|_{E^w} = \|wx\|_E$.

Further, let $\widehat{E} = K_E(\overline{L}_\infty)$, where $K_E(\cdot)$ is interpolation functor, generated by the parameter E, and $\overline{L}_\infty$ be the Banach pair of the spaces $L_\infty(R_+)$, $L_\infty^{1/t}(R_+)$. As it is known, $\|x\|_{\widehat{E}} = \|\widehat{x}\|_E$, where $\widehat{x}$ is the least concave majorant of the function $|x|$, $\widehat{E}$ is the parameter, $\widehat{E} \subset E$, $K_E(\cdot) = K_{\widehat{E}}(\cdot)$.

We describe our basic construction (naturally introduced in [4] for the lower estimates of the functions from $L_1 + L_\infty$ under the action of linear continuous operators from the pair (L_1, L_∞) into arbitrary Banach pair $\overline{B}$). For each parameter of K-method for the space E we assign the set $[E]$ of all functions $x \in \Sigma(\overline{L}_\infty)$ such that

$$\|x\|_{[E]} = \sup_{\|v\|_{D_1} \le 1} \|t \int_0^{1/t} \widehat{x}(v(\tau))d\tau\|_E < \infty$$

(if it is necessary, we consider that $\widehat{x}(0) = 0$).

Lemma 1. *The function $x \in [E]$ if and only if the function $t \int\limits_0^{1/t} \widehat{x}(v(\tau))d\tau$ belongs E for any $v \in D_1$.*

Proof of the nontrivial part of this fact is retrieved by assumption on the contrary.

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Let $\|v_n\|_{D_1} \le 1$ and $\|t \int\limits_0^{1/t} \widehat{x}(v_n(\tau))d\tau\|_E \ge n \cdot 2^n$. Put

$$v = \sum_{n=1}^{\infty} 2^{-n} \cdot v_n$$

(the convergence is almost everywhere pointwise). It is clear that $v \in D_1$ and $\|v\|_1 \le 1$. Since

$$\widehat{x}(v(\tau)) \ge \widehat{x}(2^{-n}v_n(\tau)) \ge 2^{-n}\widehat{x}(v(\tau)),$$

it follows that

$$\left\| t \int_0^{1/t} \widehat{x}(v(\tau))d\tau \right\|_E \ge 2^{-n} \left\| t \int_0^{1/t} \widehat{x}(v_n(\tau))d\tau \right\|_E \ge 2^{-n} \cdot n2^n = n$$

for any n, which leads to the contradiction.

Proposition. *$[E]$ is parameter of K-method of interpolation, $[E] \supset \widehat{E}$.*

Proof. Clearly, $([E], \|\cdot\|_{[E]})$ is a normed ideal space. Omitting the details, we show how one can verify the completeness of this space. Let $\sum\limits_{n=1}^{\infty} \|x_n\|_{[E]} < \infty$. Then the series $\sum\limits_{n=1}^{\infty} x_n$ converge in the Banach space $\Sigma(\overline{L}_\infty)$ to some x. We define $y_N = x - \sum\limits_{n<N} x_n = \sum\limits_{n \ge N} x_n$. For any $v \in D_1$ we have $\|v\|_1 \le 1$, $\widehat{y}_N(v(\tau)) \le \sum\limits_{n \ge N} \widehat{x}_n(v(\tau))$, therefore

$$\left\| t \int_0^{1/t} \widehat{y}_N(v(\tau))d\tau \right\|_E \le \sum_{n \ge N} \left\| t \int_0^{1/t} \widehat{x}_n(v(\tau))d\tau \right\|_E \le \sum_{n \ge N} \|x_n\|_{[E]}.$$

Taking the supremum by v, we obtain $\left\| x - \sum\limits_{n<N} x_n \right\|_{[E]} \le \sum\limits_{n \le N} \|x_n\|_{[E]}$. Therefore, the series $\sum\limits_{n=1}^{\infty} x_n$ converge in $[E]$ to x, and $[E]$ is complete.

For the proof of the inclusion $\widehat{E} \subset [E]$, note that the measure $td\tau$ is a probability measure on the interval $(0, 1/t)$. Using the fact that the non-negative function $\widehat{x}$ is concave, we have

$$\int_0^{1/t} \widehat{x}(v(\tau))td\tau \le \widehat{x}\left(\int_0^{1/t} v(\tau)t\, d\tau \right) \le \widehat{x}(\|v\|_1 t) \le max(1, \|v\|_1)\widehat{x}(t).$$

Therefore, if $x \in \widehat{E}$, i.e., $\widehat{x} \in E$, then $x \in [E]$ and $\|x\|_{[E]} \le \|x\|_{\widehat{E}}$. □

The study of the transformation $E \to [E]$ to certain extent is motivated by the following

Theorem 1. *If $\overline{B}$ is an arbitrary relatively complete K_0-abundant Banach couple, then*

$$K_E(L_1, L_\infty) \subset \mathrm{Corb}((L_1, L_\infty) \to \overline{B}; \;\; K_E(\overline{B})) \subset K_{[E]}(L_1, L_\infty).$$

.

Proof. The left inclusion is obvious. The right inclusion, such that it is upper estimate of the coorbit, substantially follows from the definition and theorem 1 of paper [4]. Additionally we note that for some pairs $\overline{B}$ (for example, for the pair of the form $(L_r^{W_0}, L_1^{W_1})$) the right inclusion, in fact, is the isomorphism. □

This simple Theorem in the situation of interpolation $(L_1, L_\infty) \to \overline{B}$ results in a simple upper estimate of the coorbits of the spaces of K-method, moreover, the form of the estimate does not depend on $\overline{B}$. Sometimes for particular pairs $\overline{B}$ this estimation is faithful; the examples of the faithful estimations are based on the Ovchinnikov theorem from [5]. Surely, for the classes of pairs $\overline{B}$, which have an effective description (or estimation) of the orbit, we are able to significantly improve of the estimation mentioned in Theorem 1. For example, V. I. Ovchinnikov in [6] (and also in [7]) have gained accurate interpolation theorems in far more general situation than $(L_1, L_\infty) \to (L_p, L_q)$ based on the fundamental theorem from [8] about the description of the orbits in the pairs of weight spaces L_p. In this paper we do not discuss these uneasy results.

From Theorem 1 the problem of "the computation" of the spaces $[E]$ occurs, i.e., the problem of their identification for specific E.

In this paper the transformation $E \to [E]$ is performed for the parameter-space of classical standard method of interpolation.

2. For $1 \le q \le \infty$, $\alpha \in R$ we denote by L^α_{q*} the space $L_q(R_+)$ with the weight $w(t) = t^{\alpha - 1/q}$,

$$\|x\|_{L^\alpha_{q*}} = \left(\int_0^\infty (t^\alpha |x(t)|)^q \frac{dt}{t}\right)^{1/q} \quad (q < \infty),$$
$$\|x\|_{L^\alpha_{\infty *}} = \operatorname{ess\,sup}_{t>0} t^\alpha |x(t)| \quad (q = \infty).$$

For $0 < \theta < 1$ the space $L^{-\theta}_{q*}$ are parameters for the classical K-method of interpolation:

$$K_{L^{-\theta}_{q*}}(\overline{B}) = K_{\widehat{L^{-\theta}_{q*}}}(\overline{B}) = \overline{B}_{\theta,q}.$$

Theorem 2. *The equality*

$$[L^{-\theta}_{q*}] = \widehat{L^{-\theta}_{p*}}, \quad 1/p = \big(1/q - \theta\big)_+$$

holds (*with the equivalent norms*; $a_+ = \max(a, 0)$).

Corollary. Let $\overline{B}$ be relatively complete K_0-abundant Banach pair. Then

$$\operatorname{Corb}((L_1, L_\infty) \to \overline{B};\ \overline{B}_{\theta,q}) \subset (L_1, L_\infty)_{\theta,p}, \quad 1/p = \big(1/q - \theta\big)_+,$$

and, in fact, for $q = \infty$ the embedding is an isomorphism.

In the proof of Theorem 2 we use the facts which deserve separate statements.

Lemma 2. *For any $p > 0$ there exists the constant c_p such that*

$$\sum_{n\in Z} 2^n y_n \le \sum_{n\in Z} 2^n \left(\sum_{m\ge n} y^p_m\right)^{1/p} \le c_p \sum_{n\in Z} 2^n y_n$$

for any sequence $(y_n)_{n\in Z}$ of non-negative numbers.

Proof. The left inequality is trivial, and the right one follows from the well-known inequalities. Let us explain it.

If $p \ge 1$, then $\sum\limits_{m\ge n} y^p_m \le \Big(\sum_{m\ge n} y_m\Big)^p$, therefore

$$\sum_{n\in Z} 2^n \left(\sum_{m\ge n} y^p_m\right)^{1/p} \le \sum_{n\in Z} 2^n \sum_{m\ge n} y_m = \sum_{m\in Z} y_m \sum_{n\le m} 2^n = 2 \sum_{m\in Z} y_m 2^m.$$

Let now $p < 1$. For non-negative functions f, which are measurable on R_+, the inequality

$$\|H_2 f\|_{L_{1/p}(R_+)} \le \frac{1}{p} \|f\|_{L_{1/p}(R_+)}$$

holds, where H_2 is the Hardy operator: $(H_2 f)(t) = \int\limits_t^\infty \frac{f(s)}{s} ds$.

In this inequality we put $f(t) = \sum\limits_{n \in Z} y_n^p \chi_n(t)$, where χ_n is the characteristic function of the interval $[2^n, 2^{n+1})$.

We have

$$\|H_2 f\|^{1/p}_{L_{1/p(R_+)}} = \sum_{n \in Z} \int_{2^{n-1}}^{2^n} \left(\int_t^\infty f(s) \frac{ds}{s} \right)^{1/p} dt \geq \sum_{n \in Z} \int_{2^{n-1}}^{2^n} \left(\int_{2^n}^\infty f(s) \frac{ds}{s} \right)^{1/p} dt$$

$$= \sum_{n \in Z} (2^n - 2^{n-1}) \Big(\int_{2^n}^\infty \sum_{m \in Z} y_m^p \chi_m(s) \frac{ds}{s} \Big)^{1/p} = \frac{1}{2} \sum_{n \in Z} 2^n \Big(\sum_{m \geq n} y_m^p \int_{2^m}^{2^{m+1}} \frac{ds}{s} \Big)^{1/p} =$$

$$= \frac{1}{2} (\ln 2)^{1/p} \sum_{n \in Z} 2^n \Big(\sum_{m \geq n} y_m^p \Big)^{1/p};$$

$$\|f\|^{1/p}_{L_{1/p}(R_+)} = \int_0^\infty \left(\sum_{n \in Z} y_n^p \chi_n(t) \right)^{1/p} dt = \sum_{n \in Z} y_n 2^n.$$

Therefore, the required inequality for $p < 1$ holds with $c_p = 2(p \ln 2)^{-1/p}$. □

Another important fact is the well-known "Resonance" theorem of Landau.

Lemma 3. *We will consider only non-negative sequences. Let $0 < p \leq \infty$. We have $\sum\limits_n a_n x_n < \infty$ for any sequence $(x_n) \in \ell_p$ if and only if $(a_n) \in \ell_{p*}$, where $1/p* = \big(1 - 1/p\big)_+$.*

Proof of Theorem 2. Consider the case $q < \infty$. The function x belongs to the space $[L_{q*}^{-\theta}]$, only if $\varphi = \widehat{x}$ belongs to this space. Further we use the function φ.

Let H_1 be the Hardy operator: $(H_1 f)(t) = \frac{1}{t} \int\limits_0^t f(s) ds$, $\varphi \in [L_{q*}^{-\theta}]$ means that $H_1(\varphi(v))\big(1/t\big) \in L_{q*}^{-\theta}$ for all $v \in D_1$ (Lemma 1), and this, in turn, means that $H_1(\varphi(v))(t) \in L_{q*}^{\theta}$ for all $v \in D_1$. Since, on one hand, $\varphi(v) \leq H_1(\varphi(v))$, and on the other hand $H_1 : L_{q*}^\theta \to L_{q*}^\theta$ with the norm $\leq \frac{1}{1-\theta}$, it follows that $H_1(\varphi(v)) \in L_{q*}^\theta \Leftrightarrow \varphi(v) \in L_{q*}^\theta$.

Therefore, we obtain $\varphi \in [L_{q*}^{-\theta}]$, only if $\int\limits_0^\infty (t^\theta \varphi(v(t)))^q \frac{dt}{t} < \infty$ for all $v \in D_1$.

For each function $v \in D_1$ we define its distribution function $w(s) = \sup\{t : v(t) > s\}$ for $s > 0$.

We have

$$\int_0^\infty (t^\theta \varphi(v(t)))^q \frac{dt}{t} = \sum_{n \in Z} \int_{\{2^n < v(t) \leq 2^{n+1}\}} (t^\theta \varphi(v(t)))^q \, \frac{dt}{t}$$

(due to the fact that $\varphi(2^n) \leq \varphi(v(t)) \leq \varphi(2^{n+1}) \leq 2\varphi(2^n)$ for $2^n < v(t) \leq 2^{n+1}$)

$$\asymp \sum_{n \in Z} \varphi^q(2^n) \int_{w(2^{n+1})}^{w(2^n)} t^{\theta q} \frac{dt}{t} \asymp \sum_{n \in Z} \varphi^q(2^n)(w_n^{\theta q} - w_{n+1}^{\theta q}),$$

where $w_n = w(2^n)$. For the sequence $(w_n)_{n \in Z}$ we can state the following:

$$0 \leq w_n \downarrow, \quad \sum_{n \in Z} 2^n w_n < \infty, \tag{A}$$

because

$$\sum_{n\in Z} 2^n w_n \le 2\sum_{n\in Z}\int_{2^{n-1}}^{2^n} w(s)ds = 2\int_0^\infty w(s)ds = 2\int_0^\infty v(t)dt < \infty.$$

It is easy to see that the mapping from D_1 into the set W of all of the sequences, satisfying conditions (A), transforming, as it is noted, the function v into the sequence (w_n), is surjective. Therefore,

$$\varphi \in [L_{q*}^{-\theta}] \Leftrightarrow \sum_{n\in Z} \varphi^q(2^n)(w_n^{\theta q} - w_{n+1}^{\theta q}) < \infty \quad \text{for all} \quad (w_n) \in W. \tag{B}$$

Put $y_n = 2^{\theta q n}(w_n^{\theta q} - w_{n+1}^{\theta q})$. We state that the mapping $\prod : (w_n) \to (y_n)$ is surjective (and even bijective) from W onto the cone $\ell_{1/\theta q}^+$ of all non-negative elements of the space $\ell_{1/\theta q}$. Indeed, let $(w_n) \in W$. Then $(y_n) = \prod((w_n)) \in \ell_{1/\theta q}$ because

$$\sum_{n\in Z} y_n^{1/\theta q} = \sum_{n\in Z} 2^n (w_n^{\theta q} - w_{n+1}^{\theta q})^{1/\theta q} \le \sum_{n\in Z} 2^n w_n < \infty.$$

Conversely, if $(y_n) \in \ell_{1/\theta q}^+$, then put $w_n = \Big(\sum\limits_{m\ge n} 2^{-\theta q m} y_m\Big)^{1/\theta q}$. Then $0 \le w_n \downarrow$ and

$$\sum_{n\in Z} 2^n w_n = \sum_{n\in Z} 2^n \Big(\sum_{m\ge n} 2^{-\theta q m} y_m\Big)^{1/\theta q} \text{ (according to Lemma 2)}$$

$$\le c_{\theta q} \sum_{n\in Z} 2^n \cdot 2^{-n} y_n^{1/\theta q} = c_{\theta q} \|(y_n)\|_{\ell_{1/\theta q}}^{1/\theta q} < \infty.$$

Therefore, $(w_n) \in W$ and, obviously, $\prod((w_n)) = (y_n)$.

From the above we obtain that equivalence (B) may we rewritten in the following form

$$\varphi \in [L_{q*}^{-\theta}] \Leftrightarrow \sum_{n\in Z} (2^{-\theta n}\varphi(2^n))^q y_n < \infty \quad \text{for all} \quad (y_n) \in \ell_{1/\theta q}^+.$$

From this by Lemma 3 we obtain

$$\varphi \in [L_{q*}^{-\theta}] \Leftrightarrow ((2^{-\theta n}\varphi(2^n))^q)_{n\in Z} \in \ell_{1/(1-\theta q)_+}.$$

Hence,

$$\varphi \in [L_{q*}^{-\theta}] \Leftrightarrow (2^{-\theta n}\varphi(2^n))_{n\in Z} \in \ell_p,$$

where

$$1/p = (1-\theta q)_+/q = \big(1/q - \theta\big)_+.$$

From the almost obvious equivalence of the values $\|2^{-\theta n}\varphi(2^n)\|_{\ell_p}$ and $\|\varphi\|_{L_{p*}^{-\theta}}$ we get

$$\varphi \in [L_{q*}^{-\theta}] \;\Leftrightarrow\; \widehat{x} = \varphi \in L_{p*}^{-\theta} \;\Leftrightarrow\; x \in L_{p*}^{-\theta},$$

and the Theorem is proved for $q < \infty$.

Consider the remaining case $q = \infty$. We have

$$x \in [L_{\infty *}^{-\theta}] \;\Leftrightarrow\; \sup_{\|v\|_1\le 1} \|H_1(\varphi(v))\|_{L_{\infty *}^{\theta}} < \infty.$$

Since $\varphi(v) \le H_1(\varphi(v))$ and $\|H_1(\varphi(v))\|_{L_{\infty *}^{\theta}} \le \frac{1}{1-\theta}\|\varphi(v)\|_{L_{\infty *}^{\theta}}$, it follows that

$$x \in [L_{\infty *}^{-\theta}] \;\Leftrightarrow\; \sup_{\|v\|_1\le 1} \|\varphi(v)\|_{L_{\infty *}^{\theta}} < \infty. \tag{C}$$

Let $x \in [L_{\infty *}^{-\theta}]$. We denote by M the supremum of (C). For each $t > 0$ we take $v_t(s) = t \cdot \chi_{(0,1/t]}(s)$. Clearly, $v_t \in D_1$ and $\|v_t\|_1 = 1$. The inequality $\varphi(v_t)\|_{L_{\infty *}^{\theta}} \le M$ holds, i.e., $\sup_{s>0} s^{\theta}\varphi(t \cdot \chi_{(0,1/t]}(s)) \le M$, which implies $t^{-\theta}\varphi(t) \le M$. Hence, $\|\varphi\|_{L_{\infty *}^{-\theta}} \le M$ and so $x \in \widehat{L_{\infty *}^{-\theta}}$.

Conversely, let $x \in \widehat{L_{\infty *}^{-\theta}}$, therefore $\varphi \in L_{\infty *}^{-\theta}$, i.e., $\varphi(t) \le Mt^{\theta}$ for all $t > 0$. For $v \in D_1$, $\|v\|_1 \le 1$ we have

$$t^{\theta}\varphi(v(t)) \le M(tv(t))^{\theta} \le M\biggl(\int_0^t v(s)ds\biggr)^{\theta} \le M.$$

Therefore, condition (C) is fulfilled and, hence, $x \in [L_{\infty *}^{-\theta}]$. □

3. As a complement to the above we describe (without proofs, which in many aspects repeat those in Item 2) the result of the transform $E \to [E]$ for the case when E is the weight L_1-space.

Consider the space L_1^w (see Item 1). This space is nontrivial parameter of K-method of interpolation only if it contains the function $\min(1, t)$, i.e., it is required that

$$\int_0^1 w(t)tdt < \infty, \quad \int_1^{\infty} w(t)dt < \infty.$$

In order to avoid the degenerate situations, we immediately assume

$$\int_0^1 w(t)dt = \infty, \quad \int_1^{\infty} w(t)t\,dt = \infty.$$

We define the sequence

$$\omega(s) = \int_0^{1/s} w(\tau)\tau d\tau \quad (s > 0), \quad N(t) = \int_0^t \omega(s)ds \quad (t \ge 0).$$

The function $N(t)$ is continuous increasing concave function, $N(0) = 0$, $N(\infty) = \infty$. Denote by M^* the function which is inverse function for N. The function M^* on $[0, \infty)$ is the Orlicz function, i.e., it is continuous, increasing, convex and

$$\lim_{t\to 0}\frac{M^*(t)}{t} = 0, \quad \lim_{t\to\infty}\frac{M^*(t)}{t} = \infty.$$

Let us consider complementary function M^* for the function M; this Orlicz function is defined by the formula

$$M(t) = \sup_{s\ge 0}(ts - M^*(s)).$$

Recall the Orlicz definition of BIS L_M. The elements of this space are the measurable functions f on R_+ such that

$$\|f\|_{L_M} = \inf\biggl\{\lambda > 0 : \int_0^{\infty} M\biggl(\frac{|f(t)|}{\lambda}\biggr)dt \le 1\biggr\} < \infty.$$

The space L_M takes part in our description of the space $[L_1^w]$.

Theorem 3. *$x \in [L_1^w]$, if and only if $\widehat{x}(0+) = 0$ and the function $\frac{\widehat{x}(2t)-\widehat{x}(t)}{t}$ belongs to the Orlicz space L_M.*

For the description of the space $[L_1^w]$ we are also able to use the derivative of absolutely continuous function $\widehat{x}$: $x \in [L_1^w]$ if and only if $\widehat{x}(0+) = 0$ and the derivative $\widehat{x}'$ belongs to the Orlicz space L_M.

In conclusion we give a simple example. As it is well-known, $K(t, f; L_1, L_\infty) = K(t) = \int\limits_0^t f^*(s)ds$, where f^* is the permutation of the function $|f|$ in the non-decreasing order, $K(t)$ is concave function, $\widehat{K}(t) = K(t)$. The condition $K(0+) = 0$ is fulfilled automatically. We have

$$f \in K_{[L_1^w]}(L_1, L_\infty) \Leftrightarrow K(t) \in [L_1^w] \Leftrightarrow K'(t) \in L_M \Leftrightarrow f^* \in L_M \Leftrightarrow f \in L_M.$$

Therefore,

$$K_{[L_1^w]}(L_1, L_\infty) = L_M.$$

Note, that the problem of "the computation" of the spaces $[E]$ at least for some E, which differs from the spaces considered in this paper, remains unstudied.

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Investigation of Boundary-Value Problem for Slow Flow of a Sphere by Viscous Non-Isothermal Gas

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Abstract—We obtain a solution to a boundary-value problem of a flow of spherical form particle for stationary system of equations of viscous non-isothermal gaseous medium including the Stokes equation, heat conductivity equation, and state equation with account taken of dependence of viscosity, heat conductivity, and density of gaseous medium on temperature.

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Introduction. The mathematical theory of gaseous medium flow is a wide and quickly developing part of the gas dynamics. The most interesting theme is the system of Navier–Stokes equations, which expresses the conservation lows of impulse and mass. In the vector form they are written as follows:

$$\frac{\partial \mathbf{V}}{\partial t} + \big(\mathbf{V}\cdot\nabla\big)\mathbf{V} = \nu\triangle\mathbf{V} - \frac{1}{\rho}\nabla P - \mathbf{f},\ \ \frac{\partial \mathbf{V}}{\partial t} + \nabla\cdot\rho\mathbf{V} = 0, \tag{1}$$

where $\mathbf{V} = \big(V_1, V_2, \dots, V_n\big)$ is the vector field of rates, t is the time, ∇ and $\triangle$ are the nabla and Laplace operators, ρ is the density, P is the pressure, ν is the coefficient of kinematic viscosity, $\mathbf{f}$ is the vector field of mass forces. Unknowns P and $\mathbf{V}$ are functions of time t and coordinate $x \in \Omega$, where $\Omega \in \mathbb{R}^n$, $n = 2, 3$, is a two- or three-dimensional domain, in which the medium moves.

One of the most disagreeable properties of the system of Navier–Stokes equations is its nonlinearity due to presence of a convective member of acceleration $\big(\mathbf{V}\cdot\nabla\big)\mathbf{V}$ in the left-hand side of Eq. (1). In addition, usually the edge conditions for the Navier–Stokes equations, describing flows of a concrete viscous medium, are nonlinear. Taking into account these facts, in the gas dynamics one has elaborated approximate methods, which allow one to simplify the system of hydrodynamic equations in one way or another and adjust it to the character of separate types of concrete physical problems. There exists a wide class of hydrodynamic flows, in which one can neglect the nonlinear term. In the scientific literature such equations are said to be the Navier–Stokes equations linearized with respect to the rate.

Investigating a stationary system of equations of a viscous non-isothermal gaseous medium we use the term a "relative drop of temperature", which equals quotient of difference between the mean temperature of particle T_S surface and the temperature of domain far from it T_∞ to the latter. A relative drop of temperature is assumed to be small, if the inequality is fulfilled $(T_S - T_\infty)/T_\infty \ll 1$. Provided this inequality, coefficients of viscosity, heat conductivity, and density of medium can be assumed to be constants, and the viscous medium is said to be isothermal. If $(T_S - T_\infty)/T_\infty \sim O(1)$, then the relative drop of temperature is assumed to be significant. In this case, it is necessary to take into account the dependence of indicated coefficients on temperature, that essentially complicates the analysis of system of equations, and the viscous medium is said to be non-isothermal.

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At the present time, the motion of particles of spherical form was investigated in detail in the isothermal case only (for example, [1], Chap. XI and [2], Chap. 4), in which the analysis of system of the Navier–Stokes equations is essentially simplified and solutions are well-studied unlike the non-isothermal case ([3–6]). In the present paper, we investigate crawl flows of viscous non-isothermal gas. With a definite type of search of solutions of mass rate components and with feasible simplifications from the physical point of view, a solution to a system of Navier–Stokes equations linearized with respect to rate is reduced to the solution of a fourth-order ordinary differential equation with an isolated singular point with the help of functional series of a special form.

Problem definition. Main equations and edge conditions. We consider the classical problem of flow of a solid non-uniformly heated particle of spherical form of radius R by a flatly parallel flow of gas with the rate $\mathbf{U}_\infty$ ($\mathbf{U}_\infty \| Oz$), but not with small relative temperature drops, as in [2] (P. 144), and with significant temperature drops. The description of flow is realized in the spherical system of coordinates (r, φ, θ) connected with the mass center of particle. We obtained an axis-symmetric (independent of the coordinate φ) solution to the boundary-value problem for the system of stationary gas dynamics equations describing the vector field $\mathbf{U}(x) = (U_1(x), U_2(x), U_3(x))$. We found functions $P(x)$, $T(x)$ in the domain $x \in \Omega_g = \mathbb{R}^3 \setminus \Omega_p$, where Ω_p is a spherical domain centered at zero, and also the function $T_p(x)$, $x \in \Omega_p$. Here $\mathbf{U}(x)$ is the field of rates, $P(x)$ and $T(x)$ are distributions of pressure and temperature in the outer flow, and $T_p(x)$ is the distribution of temperature inside the particle. The indicated system of equations has the following form:

$$\nabla_k P = \sum_{j=1}^{3} \nabla_j \left[\mu(\nabla_j U_k + \nabla_k U_j - \frac{2}{3}\delta_{jk}(\nabla, \mathbf{U})) \right], \quad k = 1, 2, 3, \tag{2}$$

$$\big(\nabla, \rho \mathbf{U}\big) = 0, \tag{3}$$

$$\big(\nabla, \lambda_g \nabla T_g\big) = 0, \tag{4}$$

$$\big(\nabla, \lambda_p \nabla T_p\big) = -q, \tag{5}$$

where $\nabla = (\nabla_1, \nabla_2, \nabla_3)$ is a vector differential Hamilton operator in Cartesian coordinates, $\nabla_j \equiv \partial/\partial x_j$, $q(x)$ is a given in Ω_p function defining the density of heat sources inside the particle. Hereinafter indices g and p are related, respectively, to domains Ω_g and Ω_p, i.e., to the medium and to the particle. Since μ, ρ, λ_g, and λ_p are functions with respect to desired functions $T_g(x)$ and $T_p(x)$, system of equations (2)–(5) is nonlinear in the whole. In what follows we call Eq. (2) a Navier–Stokes equation linearized with respect to rate, (3) a continuity equation, (4) and (5) heat conductivity equations, describing the distribution of temperature outside Ω_g and inside Ω_p, respectively.

The mathematical description of motion of heated particles in a viscous non-isothermal gaseous medium allows one to expand the method of solution of a linearized with respect to rate Navier–Stokes equation, elaborated for a concrete physical problem, on a wider class of problems.

In the present paper, for the solution to system of gas dynamic equations we make the following physical assumptions, which can be realized in most applied problems.

Admission 1. As in [7] (Chap. IX), we assume the power form of dependence of dynamic viscosity, heat conductivity, and density with respect to temperature:

$$\mu_g = \mu_\infty \big(T_g/T_\infty\big)^\beta, \;\; \rho_g = \rho_\infty \big(T_\infty/T_g\big), \;\; \lambda_g = \lambda_\infty \big(T_g/T_\infty\big)^\alpha, \;\; \lambda_p = \lambda_* \big(T_p/T_\infty\big)^\omega,$$

where μ_∞, ρ_∞, λ_∞, λ_*, and T_∞ are positive constants. In the indicated power dependencies, coefficients are in limits $0.5 \le \alpha, \beta \le 1$, $-1 \le \omega \le 1$. Hereinafter the index ∞ is used for the definition of values of desired functions at infinity.

Admission 2. The heat conductivity coefficient of particle is essentially greater than the coefficient of heat conductivity of gas that takes place for most real gaseous mediums. This assumption yields that in the viscosity coefficient we can neglect the dependence with respect to the angle θ in the system "particle–gaseous medium" (we assume a weak angular asymmetry of temperature distribution) and, therefore, the viscosity is connected with the temperature $t_{g0}(r)$, only, i.e., $\mu_g(t_g(r,\theta)) \approx \mu_g(t_{g0}(r))$.

Here $t_g(r,\theta) = t_{g0}(r) + \delta t_g(r,\theta)$, where $\delta t_g(r,\theta) \ll t_{g0}(r)$, and $\delta t_g(r,\theta), t_{g0}(r)$ are defined from the solution to heat problem (4), (5). With this assumption we can consider the hydrodynamic part separately of the heat part, and the connection between them is established with the help of edge conditions.

Admission 3. A particle is formed by homogeneous and isotropic by its properties material.

In the spherical system of coordinates the system of gas dynamic equations, describing the distribution of rate and pressure outside the particle, has the following form:

$$\frac{\partial P}{\partial y} = \frac{\partial \sigma_{rr}}{\partial y} + \frac{2}{y}\sigma_{rr} + \frac{1}{y}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\cot\theta}{y}\sigma_{r\theta} - \frac{\sigma_{\theta\theta} - \sigma_{\varphi\varphi}}{y}, \tag{6}$$

$$\frac{1}{y}\frac{\partial P}{\partial \theta} = \frac{\partial \sigma_{r\theta}}{\partial y} + \frac{3}{y}\sigma_{r\theta} + \frac{1}{y}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\cot\theta}{y}(\sigma_{\theta\theta} - \sigma_{\varphi\varphi}), \tag{7}$$

$$\frac{1}{y^2}\frac{\partial}{\partial y}\big(y^2 \rho U_r\big) + \frac{1}{y\sin\theta}\frac{\partial}{\partial \theta}\big(\sin\theta\,\rho U_\theta\big) = 0, \tag{8}$$

and heat conductivity equations, describing distributions of temperature outside and inside the particle, taking into account Assumption 1, have the form

$$\frac{1}{y^2}\frac{\partial}{\partial y}\left(y^2 \frac{\partial t_g^{1+\alpha}}{\partial y}\right) + \frac{1}{y^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial t_g^{1+\alpha}}{\partial \theta}\right) = 0, \tag{9}$$

$$\frac{1}{y^2}\frac{\partial}{\partial y}\left(y^2 \frac{\partial t_p^{1+\omega}}{\partial y}\right) + \frac{1}{y^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial t_p^{1+\omega}}{\partial \theta}\right) = -\frac{R^2(1+\omega)}{\lambda_* T_\infty} q, \tag{10}$$

where $y = r/R$, σ_{rr}, $\sigma_{r\theta}$, $\sigma_{\theta\theta}$ and $\sigma_{\varphi\varphi}$ are components of tensions tensor in the spherical system of coordinates defined by equalities ([8], Chap. II, P. 70)

$$\sigma_{rr} = \mu\left(2\frac{\partial U_r}{\partial y} - \frac{2}{3}\operatorname{div}\mathbf{U}\right), \quad \sigma_{\theta\theta} = \mu\left(\frac{2}{y}\frac{\partial U_\theta}{\partial \theta} + \frac{2}{y}U_r - \frac{2}{3}\operatorname{div}\mathbf{U}\right),$$

$$\sigma_{\varphi\varphi} = \mu\left(\frac{2}{y}U_r + \frac{2}{y}\cot\theta\, U_\theta - \frac{2}{3}\operatorname{div}\mathbf{U}\right), \quad \sigma_{r\theta} = \mu\left(\frac{\partial U_\theta}{\partial y} + \frac{1}{y}\frac{\partial U_r}{\partial \theta} - \frac{U_\theta}{y}\right).$$

We assume that on the particle surface (with $r = R$) the adhesion condition for normal and tangent components of mass rate $\mathbf{U}$ is fulfilled. In addition, the equality of temperature and the continuity of radial heat flows take place taking into account emanations on the particle surface, and the standard conditions also hold true as $y \to \infty$ and $y \to 0$. Therefore the system of gas dynamic equations (6)–(10) should be solved with the following boundary conditions:

$$\lim_{y\to 1} U_r(y,\theta) = \lim_{y\to 1} U_\theta(y,\theta) = 0, \quad \lim_{y\to 1} T_g(y,\theta) = \lim_{y\to 1} T_p(y,\theta), \tag{11}$$

$$\lim_{y\to 1} \lambda_g T_g \frac{\partial T_g}{\partial y} = \lim_{y\to 1}\left(\lambda_p \frac{\partial T_p}{\partial y} + R\sigma_0\sigma_1(T_p^4 - T_\infty^4)\right), \tag{12}$$

$$\lim_{y\to\infty} \mathbf{U}(y,\theta) = U_\infty\cos\theta\,\boldsymbol{e}_r - U_\infty\sin\theta\,\boldsymbol{e}_\theta, \quad \lim_{y\to\infty} P = P_\infty, \quad \lim_{y\to\infty} T_g = T_\infty, \quad \lim_{y\to 0}|T_p| < \infty, \tag{13}$$

where σ_0 is the Stefan–Boltzmann constant, σ is the integral degree of solid blackness, $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$ are base unit vectors of spherical system of coordinates.

We will seek components of mass rate $U_r(y,\theta)$, $U_\theta(y,\theta)$ and pressure $P(y,\theta)$ in the form of decomposition by the Legendre and Gegenbauer polynomials. They are necessary for finding the common force, acting to the particle, which is defined ([8], Chap. II, P. 70) by integrating of the tensions tensor with respect to the particle surface S with $r = R$,

$$\lim_{r\to R} F_z(r) = \lim_{r\to R}\int_S (-P(r/R,\theta)\cos\theta + \sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta) r^2 \sin\theta\, d\theta\, d\varphi. \tag{14}$$

Using the Legendre and Gegenbauer polynomials properties, we see that this force is defined by the first members of decompositions only ([2], Chap. 4, P. 156). Therefore we assume

$$U_r(y,\theta) = U_\infty G(y)\cos\theta,\ \ U_\theta(y,\theta) = -U_\infty g(y)\sin\theta, \tag{15}$$

where $G(y)$ and $g(y)$ are arbitrary functions depending on the radial component.

The connection between functions $G(y)$ and $g(y)$ can be found from continuity equation (8) taking into account the dependence of density of gaseous medium on temperature in Assumption 1. It has the following form:

$$g(y) = G(y) + \frac{y}{2}(G'(y) - f(y)G(y)),\ \ f(y) = \frac{1}{t_{g0}(y)}\frac{dt_{g0}(y)}{dy}. \tag{16}$$

We will find temperature fields outside and inside the particle. For that it is necessary to solve Eqs. (9)–(10). Taking into account Assumption 2, in what follows we need functions $t_{g0}(y)$ and $t_{p0}(y)$, only, which are described by equations

$$\frac{1}{y^2}\frac{d}{dy}\left(y^2\frac{d\Phi_1(y)}{dy}\right) = 0, \tag{17}$$

$$\frac{1}{y^2}\frac{d}{dy}\left(y^2\frac{d\Phi_2(y)}{dy}\right) = f(y), \tag{18}$$

where $\Phi_1(y) = t_{g0}^{(1+\alpha)}(y)$, $\Phi_2(y) = t_{p0}^{(1+\omega)}(y)$, $f(y) = -\frac{R^2(1+\omega)}{2\lambda_* T_\infty}\int\limits_{-1}^{+1} q(y,x)dx$, $x=\cos\theta$.

Integrating Eqs. (17) and (18), we obtain the following solutions, which satisfy boundary conditions as $y\to\infty$ and $y\to 0$ (see (13))

$$t_{g0}(y) = \big(1+\Gamma_0/y\big)^{1/(1+\alpha)}, \tag{19}$$

$$t_{p0}(y) = \left(B_0 + \frac{D_0}{y} + \frac{1}{y}\int_y^1 \psi_0(\xi)\,d\xi - \int_y^1 \frac{\psi_0(\xi)}{\xi}d\xi\right)^{1/(1+\omega)},$$

$$\psi_0(y) = -\frac{R^2}{2\lambda_*}\frac{1+\omega}{T_\infty}y^2\int_{-1}^1 q(y,x)\,dx,\ \ D_0 = -\int_0^1 \psi_0(\xi)d\xi,$$

where constants Γ_0 and B_0 are defined from boundary conditions on the particle surface (11), (12).

If we denote by T_S the mean value of temperature of particle surface ($T_S = t_{pS}/T_\infty,\ t_{pS} = t_{p0}$ $(y=1)$), then from the second boundary condition (11) we have

$$\Gamma_0 = \big(T_S/T_\infty\big)^{1+\alpha} - 1, \tag{20}$$

the value T_S is defined from boundary condition (12)

$$\frac{\ell(1)}{1+\alpha}\frac{\lambda_{gS}}{\lambda_{pS}}t_{gS} = \frac{R^2}{3\lambda_{pS}T_{g\infty}}J_0 - \sigma_0\sigma_1\frac{RT_\infty^3}{\lambda_{pS}}(t_{pS}^4 - 1), \tag{21}$$

in which

$$\lambda_{gS} = \lambda_\infty t_{gS}^\alpha,\ \ \lambda_{pS} = \lambda_* t_{pS}^\omega,\ \ t_{gS} = t_{g0}(y=1),$$

$$T_{gS} = t_{gS}T_\infty,\ \ \ell(1) = \frac{\Gamma_0}{1+\Gamma_0},\ \ J_0 = \frac{3}{4\pi R^3}\int_V q(x)dV,$$

and the integration is fulfilled on the whole volume of particle.

Taking into account (19) and Assumption 2, the dependence of dynamic viscosity on temperature takes the form

$$\mu_g(y,\theta) = \mu_\infty(1+\Gamma_0/y)^{\beta/(1+\alpha)}. \tag{22}$$

As in [3], differentiating (6) with respect to the variable θ, and (7) with respect to y, substituting (15) in (6), (7) and taking into account (16) and (22), after elementary transformations for the function $G(y)$ on the interval $y \in [1, \infty)$ we obtain the following homogeneous differential equation of the fourth order:

$$\frac{\mathrm{d}^4G(y)}{dy^4} + \frac{1}{y}(8+\alpha_1\ell(y))\frac{d^3G(y)}{dy^3} + \frac{1}{y^2}(8+\alpha_2\ell(y)+\alpha_3\ell^2(y))\frac{d^2G(y)}{dy^2}$$
$$+\frac{1}{y^3}(-8+\alpha_4\ell(y)+\alpha_5\ell^2(y)+\alpha_6\ell^3(y))\frac{dG(y)}{dy} + \frac{1}{y^4}(\alpha_7\ell^2(y)+\alpha_8\ell^3(y)+\alpha_6\ell^4(y))G(y) = 0, \quad (23)$$

in which

$$\alpha_2 = -\frac{8\beta}{1+\alpha},\ \alpha_3 = \frac{\beta^2-3\beta-\alpha\beta+3+3\alpha}{(1+\alpha)^2},\ \alpha_4 = 2\frac{\beta-1}{\alpha+1},\ \alpha_5 = 2\frac{\beta^2+\beta-\alpha\beta-3\alpha-3}{(1+\alpha)^2},$$

$$\alpha_1 = \frac{1-2\beta}{1+\alpha},\ \alpha_6 = \frac{6+12\alpha+6\alpha^2+\beta^2-5\beta-5\alpha\beta}{(1+\alpha)^3},$$

$$\alpha_7 = 2\frac{2+2\alpha-\beta}{(1+\alpha)^2}, \alpha_8 = -2\alpha_6,\ \ell(y) = \frac{\Gamma_0}{y+\Gamma_0},$$

with edge conditions

$$G(1) = F_1,\ \lim_{y\to\infty} G(y) = 1,\ g(1) = F_2,\ \lim_{y\to\infty} g(y) = 1, \quad (24)$$

where F_1 and F_2 are constants, whose form is defined by the concrete physical problem. For example, in the case of classical Stokes problem we have $F_1 = F_2 = 0$, that corresponds to the adhesion condition on the sphere surface of radius R.

We will seek the solution to Eq. (23) in the form of the following functional series of a special form:

$$G(y) = y^\rho \sum_{n=0}^{\infty} C_n\ell^n(y),\ C_0 \neq 0. \quad (25)$$

Calculating derivatives, we obtain

$$G'(y) = -y^{\rho-1}\sum_{n=0}^{\infty}[(n-\rho)C_n - (n-1)C_{n-1}]\ell^n(y),$$

$$G''(y) = y^{\rho-2}\sum_{n=0}^{\infty}[(n-\rho)(n-\rho+1)C_n - 2(n-\rho)(n-1)C_{n-1} + (n-1)(n-2)\,C_{n-2}]\ell^n(y),$$

$$G'''(y) = -y^{\rho-3}\sum_{n=0}^{\infty}[(n-\rho)(n-\rho+1)(n-\rho+2)C_n - 3(n-\rho)(n-\rho+1)(n-1)C_{n-1}$$
$$+3(n-\rho)(n-1)(n-2)C_{n-2} + (n-1)(n-2)(n-3)C_{n-3}]\ell^n(y),$$

$$G^{IV}(y) = y^{\rho-4}\sum_{n=0}^{\infty}[(n-\rho)(n-\rho+1)(n-\rho+2)(n-\rho+3)C_n - 4(n-\rho)(n-\rho+1)$$
$$\times(n-\rho+2)(n-1)C_{n-1} + 6(n-\rho)(n-\rho+1)(n-1)(n-2)C_{n-2}$$
$$-\ 4(n-\rho)(n-1)(n-2)(n-3)C_{n-3} + (n-1)(n-2)(n-3)(n-4)C_{n-4}]\ell^n(y).$$

Substituting series (25) in (23) and equating coefficients of y^ρ, we obtain the defining equation

$$\rho(\rho+3)(\rho+1)(\rho-2)=0,$$

having roots $\rho_1=-3$, $\rho_2=-1$, $\rho_3=0$, and $\rho_4=2$. We note that the difference of roots equals an integer. Therefore, in accordance with the general theory of solution of differential equations in the form of generalized power series (the Frobenius method) in all other solutions, except the first one, corresponding to $\rho_1=-3$, an additional summand appears, which contains a coefficient $\ln y$ multiplied by the first solution ([9], Chap. IV).

The solution

$$G_1(y)=\frac{1}{y^3}\sum_{n=0}^{\infty}C_{n,1}\ell^n(y) \tag{26}$$

corresponds to the greatest root by module.

Substituting (26) into (23) and using the method of indefinite coefficients, we obtain the following recurrent formula for finding coefficients $C_{n,1}$:

$$\begin{aligned}C_{0,1}=1,\ \ C_{n,1}=\frac{1}{n(n+2)(n+3)(n+5)}\Big\{&(n+2)\\ &\times[4(n-1)(n^2+4n+1)+\alpha_1(n+3)(n+4)+\alpha_4-\alpha_2(n+3)]C_{n-1,1}\\ &-[2(n-1)(n-2)(3n^2+9n+4)+3\alpha_1(n-2)(n+2)(n+3)\\ &-2\alpha_2(n+2)(n-2)+\alpha_3(n+1)(n+2)+\alpha_4(n-2)-\alpha_5(n+1)+\alpha_7]C_{n-2,1}\\ &+[4(n+1)(n-1)(n-2)(n-3)+3\alpha_1(n-2)(n+2)(n-3)-\alpha_2(n-3)(n-2)\\ &+2\alpha_3(n+1)(n-3)-\alpha_5(n-3)+\alpha_6 n-\alpha_8]C_{n-3,1}\\ &-(n-3)[(n-1)(n-2)(n-4)+\alpha_1(n-2)(n-4)+\alpha_3(n-4)+\alpha_6]C_{n-4,1}\Big\}.\end{aligned}$$

Hereinafter we assume $C_{n,k}=0$, if $n<0$.

We will seek the second solution to Eq. (23), which is linearly independent with the solution $G_1(y)$ and corresponds to the root $\rho_2=-1$, in the form

$$G_2(y)=\frac{1}{y}\sum_{n=0}^{\infty}C_{n,2}\ell^n(y)+\frac{\omega_1\ln y}{y^3}\sum_{n=0}^{\infty}C_{n,1}\ell^n(y).$$

Analogously, by the method of indefinite coefficients we obtain the following recurrent formula for finding coefficients $C_{n,2}$:

$$C_{0,2}=1,\ C_{1,2}=-\frac{1}{8}(6\alpha_1-2\alpha_2+\alpha_4),\ C_{2,2}=1,$$

$$\begin{aligned}C_{n,2}=\frac{1}{n(n-2)(n+3)(n+1)}\Big\{&n[4(n-1)(n^2-3)+\alpha_1(n+1)(n+2)+\alpha_4\\ &-\alpha_2(n+1)]C_{n-1,2}-[2(n-1)(n-2)(3n^2-3n-2)+3\alpha_1 n(n-2)(n+1)-2\alpha_2 n(n-2)\\ &+\alpha_3 n(n-1)+\alpha_4(n-2)-\alpha_5(n-1)+\alpha_7]C_{n-2,2}+[4(n-1)^2(n-2)(n-3)\\ &+3\alpha_1 n(n-2)(n-3)-\alpha_2(n-3)(n-2)+2\alpha_3(n-1)(n-3)-\alpha_5(n-3)-\alpha_8\\ &+\alpha_6(n-2)]C_{n-3,2}-(n-3)[(n-1)(n-2)(n-4)+\alpha_1(n-2)(n-4)\\ &+\alpha_3(n-4)+\alpha_6]C_{n-4,2}+\frac{\omega_1}{\Gamma_0^2}\sum_{k=0}^{n-2}(n-k-1)\Delta_k\Big\},\end{aligned}$$

where

$$\frac{\omega_1}{\Gamma_0^2} = \frac{1}{30}[2\alpha_3 - \alpha_5 + \alpha_7 - 2(4 + 12\alpha_1 - 3\alpha_2 + \alpha_4)C_{1,2}],$$

$$\begin{aligned}\Delta_k = {} & (4k^3 + 30k^2 + 62k + 30)C_{k,1} \\ & - [12(k^2-1)(k+3) + \alpha_4 + \alpha_1(3k^2+18k+26) - \alpha_2(2k+5)]C_{k-1,1} \\ & + [6(k-1)(k-2)(2k+3) - 2\alpha_2(k-2) - \alpha_5 + \alpha_3(2k+3) + 3\alpha_1(k-2)(2k+5)]C_{k-2,1} \\ & - [4(k-1)(k-2)(k-3) + 3\alpha_1(k-2)(k-3) + \alpha_6 + 2\alpha_3(k-3)]C_{k-3,1}.\end{aligned}$$

We will seek the third solution to Eq. (23), which is linearly independent with solutions $G_1(y)$, $G_2(y)$ and corresponds to the root $\rho_3 = 0$, in the form

$$G_3(y) = \sum_{n=0}^{\infty} C_{n,3}\ell^n(y) + \frac{\omega_2 \ln y}{y^3}\sum_{n=0}^{\infty} C_{n,1}\ell^n(y) + \frac{\omega_0 \ln y}{y}\sum_{n=0}^{\infty} C_{n,2}\ell^n(y),$$

where coefficients $C_n^{(3)}$ $(n \geq 4)$ are defined from the recurrent formula

$$C_{0,3} = 1,\ \ C_{1,3} = 0,\ \ C_{2,3} = \frac{\alpha_7}{8},\ \ C_{3,3} = 1,\ \ \omega_0 = 0,$$

$$\begin{aligned}C_{n,3} = {} & \frac{1}{n(n+2)(n-3)(n-1)}\Big\{(n-1)[4(n-1)(n^2-2n-2) + \alpha_1 n(n+1) + \alpha_4 \\ & - \alpha_2 n]C_{n-1,3} - [2(n-1)(n-2)(3n^2-9n+4) + 3\alpha_1 n(n-2)(n-1) + (n-1)(n-2) \\ & \times (\alpha_3 - 2\alpha_2) + (n-2)(\alpha_4 - \alpha_5) + \alpha_7]C_{n-2,3} + [4(n-1)(n-2)^2(n-3) \\ & + 3\alpha_1(n-1)(n-2)(n-3) + (n-3)(n-2)(2\alpha_3 - \alpha_2) + (n-3)(\alpha_6 - \alpha_5) + \alpha_7]\, C_{n-3,3} \\ & - (n-3)[(n-1)(n-2)(n-4) + \alpha_1(n-2)(n-4) + \alpha_3(n-4) + \alpha_6]C_{n-4,3} \\ & + \frac{1}{60}\left[\alpha_8 - \frac{\alpha_7}{4}(8 + 12\alpha_1 - 3\alpha_2 + \alpha_4)\right]\sum_{k=0}^{n-3}(n-k-2)(n-k-1)\Delta_k.\end{aligned}$$

In accordance with the Frobenius method, the fourth solution $G_0(y)$ to Eq. (23), which corresponds to root $\rho_4 = 2$, should be seek in the form

$$\begin{aligned}G_0(y) = {} & y^2\sum_{n=0}^{\infty} C_{n,4}\ell^n(y) + \frac{\varsigma_0}{y^3}\ln y\sum_{n=0}^{\infty} C_{n,1}\ell^n(y) \\ & + \frac{\varsigma_1}{y}\ln y\sum_{n=0}^{\infty} C_{n,2}\ell^n(y) + \varsigma_2 \ln y\sum_{n=0}^{\infty} C_{n,3}\ell^n(y),\ \ C_{0,4} = 1.\end{aligned}$$

Since this solution does not satisfy the edge condition $\lim\limits_{y\to\infty} G(y) = 1$, we do not present its explicit form here.

We note that the choice of constants $C_{0,1}$, $C_{0,2}$, and $C_{0,3}$ is realized so that the functions $G_1(y)$, $G_2(y)$, and $G_3(y)$ tend to the corresponding functions for sphere with small relative drops of temperature ([2], P. 144; [8], P. 83), i.e., that with $\Gamma_0 \to 0$

$$G_1(y) \to 1/y^3,\ \ G_2(y) \to 1/y,\ \ G_3(y) = 1.$$

Taking into account the inequality

$$\ell(y) = \frac{\Gamma_0}{y + \Gamma_0} < 1,$$

we assume that series, defining functions $G_i(y)$, $i=1,2,3$, uniformly converge with $y\geq 1$ and are bounded functions, which can be differentiated necessary number of times. Further we need derivatives till the third order. We fulfill necessary calculations and obtain

$$G_1'(y)=-\frac{1}{y^4}\sum_{n=0}^{\infty}[(n+3)C_n^{(1)}-(n-1)C_{n-1,1}]\ell^n(y),$$

$$G_1''(y)=\frac{1}{y^5}\sum_{n=0}^{\infty}[(n+3)(n+4)C_n^{(1)}-2(n+3)(n-1)C_{n-1,1}+(n-1)(n-2)C_{n-2,1}]\ell^n(y),$$

$$G_1'''(y)=-\frac{1}{y^6}\sum_{n=0}^{\infty}[(n+3)(n+4)(n+5)C_{n,1}-3(n+3)(n+4)(n-1)C_{n-1,1}$$
$$+3(n+3)(n-1)(n-2)C_{n-2,1}+(n-1)(n-2)(n-3)C_{n-3,1}]\,\ell^n(y),$$

$$G_2'(y)=-\frac{1}{y^2}\sum_{n=0}^{\infty}[(n+1)C_{n,2}-(n-1)C_{n-1,2}]\ell^n(y)+\frac{\omega_1}{y^4}\ln y\sum_{n=0}^{\infty}C_{n,1}\ell^n(y)+\omega_1\ln y\,C_1^I(y),$$

$$G_2''(y)=\frac{1}{y^3}\sum_{n=0}^{\infty}[(n+1)(n+2)\,C_{n,2}-2(n+1)(n-1)C_{n-1,2}+(n-1)(n-2)\,C_{n-2,2}]\ell^n(y)$$
$$-\frac{\omega_1}{y^5}\sum_{n=0}^{\infty}[(2n+7)C_{n,1}-2(n-1)C_{n-1,1}]+\omega_1\ln y\,G_1''(y),$$

$$G_2'''(y)=-\frac{1}{y^4}\sum_{n=0}^{\infty}[(n+1)(n+2)(n+3)C_{n,2}-3(n+1)(n+2)(n-1)C_{n-1,2}$$
$$+3(n+1)(n-1)(n-2)C_{n-2,2}+(n-1)(n-2)(n-3)C_{n-3,2}]\ell^n(y)$$
$$+\frac{\omega_1}{y^6}\sum_{n=0}^{\infty}[(3n^2+24n+47)C_{n,1}-3(n-1)(2n+7)\,C_{n-1,1}+3(n-1)(n-2)\,C_{n-2,1}]\ell^n(y)$$
$$+\omega_1\ln y\,G_1''(y),$$

$$G_3'(y)=-\frac{1}{y}\sum_{n=0}^{\infty}[nC_{n,3}-(n-1)C_{n-1,3}]\ell^n(y)+\frac{\omega_2}{y^4}\ln y\,\sum_{n=0}^{\infty}C_{n,1}\ell^n(y)+\omega_2\ln y\,G_1'(y),$$

$$G_3''(y)=\frac{1}{y^2}\sum_{n=0}^{\infty}[n(n+1)C_{n,3}-2n(n-1)C_{n-1,3}+(n-1)(n-2)C_{n-2,3}]\ell^n(y)$$
$$-\frac{\omega_2}{y^5}\sum_{n=0}^{\infty}[(2n+7)C_{n,1}-2(n-1)C_{n-1,1}]+\omega_2\ln y\,G_1''(y),$$

$$G_3'''(y)=-\frac{1}{y^3}\sum_{n=0}^{\infty}[n(n+1)(n+2)C_{n,3}-3n(n+1)(n-1)C_{n-1,3}+3n(n-1)$$
$$\times(n-2)C_{n-2,3}+(n-1)(n-2)(n-3)C_{n-3,3}]\ell^n(y)+\frac{\omega_2}{y^6}\sum_{n=0}^{\infty}[(3n^2+24n+47)C_{n,1}$$

$$- 3(n-1)(2n+7)C_{n-1,1} + 3(n-1)(n-2)C_{n-2,1}]\ell^n(y) + \omega_2 \ln y\, G_1''(y).$$

Therefore, the general solution to Eq. (23) has the form

$$G(y) = A_1G_1(y) + A_2G_2(y) + A_3G_3(y) + A_0G_0(y), \tag{27}$$

where A_1, A_2, A_3, and A_0 are arbitrary constants.

Constants A_1, A_2, A_3, and A_0 are uniquely defined from edge conditions (24). Evidently, $A_0 = 0$, $A_3 = 1$. For defining constants A_1 and A_2 we have the following linear system of equations:

$$A_1\, G_1(1) + A_2G_2(1) = F_1 - G_3(1);$$
$$A_1\, G_4(1) + A_2G_5(1) = F_2 - G_6(1),$$

where

$$G_k(y) = \left(1 + \frac{\ell(y)}{2(1+\alpha)}\right)G_{k-3}(y) + \frac{y}{2}G'_{k-3}(y), \quad k = 4, 5, 6.$$

This system has a unique solution, because its main determinant differs from zero due to the linear independence of solutions $G_1(y)$, $G_2(y)$, and $G_3(y)$. Hence, we have

$$A_1 = \frac{\begin{vmatrix} F_1 - G_3(1) & G_2(1) \\ F_2 - G_6(1) & G_5(1) \end{vmatrix}}{\begin{vmatrix} G_1(1) & G_2(1) \\ G_4(1) & G_5(1) \end{vmatrix}}, \quad A_2 = \frac{\begin{vmatrix} G_1(1) & F_1 - G_3(1) \\ G_4(1) & F_2 - G_6(1) \end{vmatrix}}{\begin{vmatrix} G_1(1) & G_2(1) \\ G_4(1) & G_5(1) \end{vmatrix}}. \tag{28}$$

As a result of the fulfilled investigation, we have proved the following

Theorem. *A function $G(y) = A_1G_1(y) + A_2G_2(y) + G_3(y)$ with coefficients, which are defined by formula* (28)*, is a unique solution to Eq.* (23)*, satisfying edge conditions* (24).

Let us turn to the definition of components of mass rate $\mathbf{U}$ and pressure P, which are necessary for finding the common force (14), acting on a non-uniformly heated particle moving in the non-isothermal gaseous medium. Taking into account (15), (16) and the proved above Theorem, we have

$$U_r(y,\theta) = U_\infty \cos\theta[A_1G_1(y) + A_2G_2(y) + G_3(y)], \tag{29}$$

$$U_\theta(y,\theta) = -U_\infty \sin\theta[A_1G_4(y) + A_2G_5(y) + G_6(y)]. \tag{30}$$

Since the explicit form of functions $U_r(y,\theta)$ and $U_\theta(y,\theta)$ is known from (29), (30), we can easily obtain from (7) expressions for the pressure field

$$P(y,\theta) = P_\infty + \frac{\mu_{e\infty}U_\infty}{R}t_{e0}^{\beta}\left(\frac{y^2}{2}G'(y) + y\left(3 + \frac{\beta-1}{2}yf(y)\right)G''(y) - \left(2 - y^2f'(y) - \frac{\beta}{2}y^2f^2(y)\right.\right.$$
$$\left.\left. + (\beta-2)yf(y)\right)G'(y) + 2\left(y^2f''(y) + yf'(y)(4+\beta yf(y)) - \frac{2}{3}f(y)\right)G(y)\right)\cos\theta,$$

$$f(y) = -\frac{\ell(y)}{(1+\alpha)y}.$$

The common force, acting on a particle, is defined by integrating of the tensions tensor with respect to its surface and in the spherical system of coordinates it is defined by formula (14). Integrating (14) by angles ($0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$), we obtain

$$\boldsymbol{F}_z = 6\pi R\mu_{e\infty}U_\infty f_\mu \boldsymbol{n}_z, \quad f_\mu = \frac{2N_2}{3N_1}, \tag{31}$$

where $\boldsymbol{n}_z$ is a unit vector at the direction of axis Oz, $N_1 = G_1(1)G_2'(1) - G_2(1)G_1'(1)$, $N_2 = G_1(1)G_3'(1) - G_3(1)G_1'(1)$.

A spherical particle, falling by gravity, acquires a constant speed $\boldsymbol{U}_p$, as soon as the gravity action is balanced by the hydrodynamic forces.

The gravity acting to the particle, taking into account the extruding force, equals

$$\boldsymbol{F}_g = (\rho_{pS} - \rho_{gS})g\frac{4}{3}\pi R^3\boldsymbol{n}_z, \tag{32}$$

where g is the acceleration of gravity, ρ_{pS} and ρ_{gS} are the densities of particle and gaseous medium, taken with the mean temperature of particle surface equal T_S.

Equating (32) to (31) and taking into account that $\boldsymbol{U} = -\boldsymbol{U}_p$, we obtain the expression for the rate of a steady fall of a solid non-uniformly heated spherical particle in the gravity field

$$\boldsymbol{U}_p = h\boldsymbol{n}_z, \;\; h = \frac{2}{9}\frac{\rho_{pS} - \rho_{gS}}{\mu_\infty f}R^2 g. \tag{33}$$

Therefore, formulas (31) and (33) allow one to estimate the force, acting on a non-uniformly heated sphere, and the rate of its gravity fall taking into account the power form of dependence of viscosity, heat conductivity, and density coefficients of gaseous medium on temperature with arbitrary relative drops of temperature between the particle surface and the domain far from it.

In the case, when the value of heating of particle surface is sufficiently small, i.e., the mean temperature of particle surface is insufficiently differs from the temperature of medium far from it ($\Gamma_0 = 0$), we can neglect the dependence of density and coefficients of molecular transfer on temperature, and then

$$G_1(1) = 1,\;\; G_1'(1) = -3,\;\; G_2(1) = 1,\;\; G_2'(1) = -1,\;\; G_3(1) = 1,\;\; G_3'(1) = 0,\;\; N_1 = 2,\;\; N_2 = 3.$$

In this case formulas (31) and (33) are reduced to the known expressions for a sphere obtained by Stokes ([2], P. 144; [8], P. 83)

$$\boldsymbol{F}_S = 6\pi R\mu_{e\infty}\boldsymbol{U}_\infty, \;\; \boldsymbol{U} = \frac{2}{9}\frac{\rho_{i\infty} - \rho_{e\infty}}{\mu_{e\infty}}R^2\boldsymbol{g}.$$

Here we note that the coefficients of molecular transfer and the density are taken with the temperature of particle surface equal the temperature of medium (in our case T_∞), i.e., these formulas are true with small relative drops of temperature.

However from (20) it follows that the constant Γ_0 depends on the value of mean temperature of particle surface T_S, which in the case of non-uniform heating of surface is defined from Eq. (21) and therefore it depends on the density of heat sources, which are non-uniformly distributed in the particle volume. Hence it follows that the functions G_1, G_2, etc also depend on density of heat sources, because these functions contain the value

$$\ell(y) = \frac{\Gamma_0}{y + \Gamma_0}.$$

The numerical analysis fulfilled with the help of above obtained formulas shows the nonlinear type of dependence of the force and rate of gravity on the mean temperature of particle surface. The obtained results for a system of gas dynamic equations allow one to describe a wide class of other physical problems, for example, the particle precipitation in the channels of different temperatures, the atmospheric sensing by a powerful laser radiation, the development of fine methods of gas cleaning of hydraulic and aerosol impurities, etc.

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Information Inequalities for Characteristics of Group-Sequential Test with Groups of Observations of Random Size

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Abstract—We consider a group-sequential test for testing a simple hypothesis against a composite one-sided alternative, which defines the following sequential statistical procedure: At each stage a random number of independent identically distributed observations (a group of observations) is observed and, based on the collected data, the decision to accept or to reject the hypothesis or to continue the observation is made. For the tests with finite number of observations, we prove the existence of the derivative of the power function and establish the information-type inequalities relating that derivative to other characteristics of the test: the average number of observations and the type I error.

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INTRODUCTION

Suppose that at any stage of a sequential experiment we observe a group of independent and identically distributed (i. i. d.) observations and the number of observations in each group is random and independent of the observations.

Let $X_1^{(\eta_1)}, X_2^{(\eta_2)}, ..., X_n^{(\eta_n)}, ...$ be a sequence of groups of observations, $X_k^{(\eta_k)} = (X_{k1}, X_{k2}, \dots, X_{k\eta_k})$ for $k = 1, 2, \dots$, where X_{kj}, $k = 1, 2, \dots$, $j = 1, 2, \dots$, are i. i. d. random variables. Let the distribution of the random variable X_{kj}, $k = 1, 2, \dots$, $j = 1, 2, \dots$, depend on an unknown parameter θ, $\theta \in \Theta$, where Θ is an open subset of the real line. We consider the problem of testing $H_0 : \theta = \theta_0$ vs $H_1 : \theta > \theta_0$, where $\theta \in \Theta$ is a fixed number.

For a measurable space $(\mathfrak{X}, \mathscr{X})$ and an integer $\eta \geq 1$, define $\mathfrak{X}^\eta$ as an η-fold Cartesian product of $\mathfrak{X}$, and $\mathscr{X}^\eta$ as a cylindrical σ-algebra in $\mathfrak{X}^\eta$. Let $\mathfrak{X}^0$ be a set consisting of a sole element denoted as "( )" (an empty "vector" with the meaning of "no observations in the group"), and $\mathscr{X}^0$ be a trivial σ-algebra in $\mathfrak{X}^0$ (this corresponds to the situation when at the moment of another observation there happen to be no objects to observe).

Let $\mathscr{G} \subseteq \{0, 1, 2, \dots\}$ be a set of admissible values of (random) sizes of groups of observations. Let $\mathscr{G}^n$ be an n-fold Cartesian product of $\mathscr{G}$. For any $\eta = (\eta_1, \dots, \eta_n) \in \mathscr{G}^n$ define $\mathfrak{X}^\eta = \mathfrak{X}^{\eta_1} \times \dots \times \mathfrak{X}^{\eta_n}$, and let $\mathscr{X}^\eta = \mathscr{X}^{\eta_1} \otimes \dots \otimes \mathscr{X}^{\eta_n}$ be the corresponding cylindrical σ-algebra in $\mathfrak{X}^\eta$. Let also $\mathscr{G}^* = \bigcup_{n\geq 1} \mathscr{G}^n$.

A group-sequential test (with groups of observations of random size) is a pair (ψ, ϕ), consisting of a (randomized) *stopping rule* ψ and a (randomized) *decision rule* ϕ, where $\psi = \{\psi_\eta : \mathfrak{X}^\eta \mapsto [0, 1],\ \eta \in \mathscr{G}^*\}$, $\phi = \{\phi_\eta : \mathfrak{X}^\eta \mapsto [0, 1],\ \eta \in \mathscr{G}^*\}$ are families of measurable functions indexed with $\eta \in \mathscr{G}^*$.

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The value of $\psi_\eta = \psi_\eta(x_1^{(\eta_1)}, x_2^{(\eta_2)}, \ldots, x_n^{(\eta_n)})$ is interpreted as the conditional probability to stop (and proceed to the decision making), given the group sizes $\eta = (\eta_1, \ldots, \eta_n) \in \mathscr{G}^n$ and the observations $(x_1^{(\eta_1)}, x_2^{(\eta_2)}, dotsc, x_n^{(\eta_n)})$ obtained up to current stage.

We assume that when the experiment stops at a stage n with groups of observations of sizes $\eta = (\eta_1, \ldots, \eta_n) \in \mathscr{G}^n$, a decision rule ϕ_η is used to make a decision. The value of $\phi_\eta(x_1^{(\eta_1)}, x_2^{(\eta_2)}, \ldots, x_n^{(\eta_n)})$ is interpreted as the conditional probability to *reject* the null hypothesis H_0, given the group sizes $\eta = (\eta_1, \ldots, \eta_n)$ and the observations $(x_1^{(\eta_1)}, x_2^{(\eta_2)}, \ldots, x_n^{(\eta_n)})$.

For any $\eta = (\eta_1, \ldots, \eta_n) \in \mathscr{G}^*$, define

$$t_\eta^\psi(x_1^{(\eta_1)}, x_2^{(\eta_2)}, \ldots, x_n^{(\eta_n)}) = (1 - \psi_{(\eta_1)}(x_1^{(\eta_1)}))(1 - \psi_{(\eta_1,\eta_2)}(x_1^{(\eta_1)}, x_2^{(\eta_2)})) \ldots (1 - \psi_{(\eta_1,\eta_2,\ldots,\eta_{n-1})}(x_1^{(\eta_1)}, x_2^{(\eta_2)}, \ldots, x_{n-1}^{(\eta_{n-1})}))$$

and

$$s_\eta^\psi(x_1^{(\eta_1)}, x_2^{(\eta_2)}, \ldots, x_n^{(\eta_n)}) = t_\eta^\psi(x_1^{(\eta_1)}, x_2^{(\eta_2)}, \ldots, x_n^{(\eta_n)})\psi_\eta(x_1^{(\eta_1)}, x_2^{(\eta_2)}, \ldots, x_n^{(\eta_n)})$$

(by definition, we suppose $t_\eta^\psi(x^{(\eta)}) \equiv 1$ for $\eta \in \mathscr{G}^1$).

Let the size of the kth group of observations be defined by a (discrete) random variable ν_k that is distributed according to a known probability mass function. $p_k(i) = P(\nu_k = i)$, $i \in \mathscr{G}$, $k = 1, 2, \ldots$ For any $\eta \in \mathscr{G}^n$ define

$$p(\eta) = P(\nu_1 = \eta_1, \ldots, \nu_n = \eta_n) = \prod_{k=1}^{n} P(\nu_k = \eta_k) = \prod_{k=1}^{n} p_k(\eta_k), \;\; n = 1, 2, \ldots$$

A stopping rule ψ generates a (discrete) random variable τ_ψ (stopping time) with a probability mass function.

$$P_\theta(\tau_\psi = n) = \sum_{\eta \in \mathscr{G}^n} p(\eta) E_\theta s_\eta^\psi. \tag{1}$$

Here and henceforth, $E_\theta(\cdot)$ denotes the expectation with respect to the distribution P_θ of observations.

In Eq. (1), we assume $s_\eta^\psi = s_\eta^\psi(X_1^{(\eta_1)}, X_2^{(\eta_2)}, \ldots, X_n^{(\eta_n)})$, in spite of its initial definition as $s_\eta^\psi = s_\eta^\psi(x_1^{(\eta_1)}, x_2^{(\eta_2)}, \ldots, x_n^{(\eta_n)})$. We will use this "duality" for any function of observations throughout the paper, using the following rule, which allows unambiguous interpretation ([1–6]). If F_η, $\eta \in \mathscr{G}^n$, is some function of observations ($F_\eta = F_\eta(x_1^{(\eta_1)}, x_2^{(\eta_2)}, \ldots, x_n^{(\eta_n)})$ or $F_\eta = F_\eta(X_1^{(\eta_1)}, X_2^{(\eta_2)}, \ldots, X_n^{(\eta_n)})$) and its arguments are omitted, then

— if F_η is under the probability or expectation sign, then $F_\eta = F_\eta(X_1^{(\eta_1)}, X_2^{(\eta_2)}, \ldots, X_n^{(\eta_n)})$,

— otherwise, $F_\eta = F_\eta(x_1^{(\eta_1)}, x_2^{(\eta_2)}, \ldots, x_n^{(\eta_n)})$.

The power function of a group-sequential test (ψ, ϕ) is defined as

$$\beta_\theta(\psi, \phi) = P_\theta \text{ (reject } H_0) = \sum_{\eta \in \mathscr{G}^*} p(\eta) E_\theta s_\eta^\psi \phi_\eta. \tag{2}$$

The type I error probability of the test (ψ, ϕ) is then $\alpha(\psi, \phi) = \beta_{\theta_0}(\psi, \phi)$.

In this paper we prove the existence, under some general conditions, of the derivative of the power function of group-sequential tests and establish the inequalities relating the derivative of the power function to the average number of observations and the type I error probability of the test. The results of this paper extend the results of section 3 in [6] to the case of group-sequential tests.

1. ASSUMPTIONS AND NOTATION

Let X_{kj} have a "density function" (a Radon–Nikodym derivative of its distribution) f_θ with respect to some σ-finite measure μ on $\mathfrak{X}$, $k = 1, 2, \dots, j = 1, 2, \dots$

By independence, given group sizes $\nu_1 = \eta_1, \dots, \nu_n = \eta_n$, $\eta = (\eta_1, \eta_2, \dots, \eta_n) \in \mathscr{G}^n$, $n \geq 1$, the random vector of observations $(X_1^{(\eta_1)}, X_2^{(\eta_2)}, \dots, X_n^{(\eta_n)})$ has a joint "density"

$$f_\theta^\eta(x_1^{(\eta_1)}, x_2^{(\eta_2)}, \dots, x_n^{(\eta_n)}) = \prod_{k=1}^{n} \prod_{j=1}^{\eta_k} f_\theta(x_{k,j})$$

with respect to a product measure

$$\mu^\eta = \mu^{\eta_1} \otimes \mu^{\eta_2} \otimes \dots \otimes \mu^{\eta_n}$$

on $\mathfrak{X}^\eta$, where

$$\mu^{\eta_k} = \mu \otimes \mu \otimes \dots \otimes \mu \, (\eta_k \text{ times}).$$

(By definition, we suppose $\prod_{j=1}^{0} (\cdot) \equiv 1$; respectively, μ^0 is a measure on $\mathfrak{X}^0$, concentrated on its sole element "( )"; let it be a probability measure.)

We will assume that the following conditions are fulfilled (when needed).

Let

$$I(\theta_0, \theta_1) = E_{\theta_0} \ln \frac{f_{\theta_0}(X_1)}{f_{\theta_1}(X_1)} \tag{3}$$

be the Kullback–Leibler information (see, e.g., [7]) in one observation for distinguishing between $\theta = \theta_0$ and $\theta = \theta_1$. (Here and henceforth, the expectation with respect to any "density function" $f(x)$, $Eg(X) = \int g(x)f(x)\,d\mu(x)$, is understood as $Eg(X) = \int g(x)f(x)I_{\{f(x)>0\}}(x)\,d\mu(x)$, so there is no need for special definition of $g(x)$ on $\{f(x) = 0\}$.)

Assumption 1 ([8], assumption 1)**.** There exists γ_1 such that

$$\limsup_{\theta \to \theta_0} I(\theta_0, \theta)/(\theta - \theta_0)^2 = \gamma_1 < \infty. \tag{4}$$

Assumption 2. There exists an integrable (with respect to μ) function $\dot{f}_{\theta_0}$ such that

$$\int |f_\theta - f_{\theta_0} - (\theta - \theta_0)\dot{f}_{\theta_0}| d\mu = o(\theta - \theta_0),$$

as $\theta \to \theta_0$.

Assumption 2 is the Fréchet differentiability of f_θ at $\theta = \theta_0$ in the space $L_1(\mu)$ of functions integrable with respect to μ (cf. similar conditions in [9, 10]).

Assumption 2 guarantees the differentiability of the power function of any test based on a fixed number of observations, as well as its differentiability under the integral sign ([6], assumption 2).

It follows from Assumption 2 that

$$E_{\theta_0} \big| \dot{f}_{\theta_0}(X_1)/f_{\theta_0}(X_1) \big| < \infty. \tag{5}$$

Equation (5) is weaker than assumption 4 in [8], where the finiteness of the Fisher information is required. In particular, if the Fisher information in one observation is finite:

$$I(\theta_0) = E_{\theta_0} \big(\dot{f}_{\theta_0}(X_1)/f_{\theta_0}(X_1) \big)^2 < \infty, \tag{6}$$

then, by the Hölder inequality, (5) is fulfilled. In turn, (6) is closely related to Assumption 1, since under rather general conditions of regularity of the statistical experiment

$$I(\theta, \theta + h) \sim I(\theta)h^2/2, \quad h \to 0.$$

Assumption 3. There exists $\delta_1 < \infty$ such that $E\nu_k < \delta_1$ for all integer $k \geq 1$.

2. MAIN RESULT

Let us prove the existence of the derivative of the power function of any test with finite number of observations and establish the inequalities relating the derivative to the average number of observations and the type I error.

The following lemma is a variant of Jensen inequality adapted to the group-sequential context.

Lemma 1 ([6], lemma 1). *Let $G : [0,\infty) \mapsto \mathbb{R} \cup \{\infty\}$ be any convex function, and let $a_\eta : \mathfrak{X}^\eta \mapsto \mathbb{R}^+$, $b_\eta : \mathfrak{X}^\eta \mapsto \mathbb{R}^+$, $\eta \in \mathscr{G}^*$, be any families of nonnegative measurable functions. Then, if*

$$0 < \sum_{\eta\in\mathscr{G}^*} p(\eta) E_{\theta_0} s_\eta^\psi a_\eta < \infty,$$

then

$$\frac{\sum\limits_{\eta\in\mathscr{G}^*} p(\eta) E_{\theta_0} s_\eta^\psi a_\eta G(b_\eta)}{\sum\limits_{\eta\in\mathscr{G}^*} p(\eta) E_{\theta_0} s_\eta^\psi a_\eta} \geq G\left(\frac{\sum\limits_{\eta\in\mathscr{G}^*} p(\eta) E_{\theta_0} s_\eta^\psi a_\eta b_\eta}{\sum\limits_{\eta\in\mathscr{G}^*} p(\eta) E_{\theta_0} s_\eta^\psi a_\eta}\right).$$

Define the information contained in a group-sequential sample obtained in accordance with the rule ψ (analog of Kullback–Leibler information), as

$$I(\theta_0,\theta;\psi) = \sum_{n=1}^{\infty} \sum_{\eta\in\mathscr{G}^n} p(\eta) E_{\theta_0} s_\eta^\psi \left(\sum_{k=1}^{n}\sum_{j=1}^{\eta_k} \ln \frac{f_{\theta_0}(X_{kj})}{f_\theta(X_{kj})}\right).$$

Applying Lemma 1 to $G(x) = -\ln(x)$, $a_\eta \equiv 1$, $b_\eta = f_\theta^\eta / f_{\theta_0}^\eta$, $\eta \in \mathscr{G}^*$, and assuming that $P_{\theta_0}(\tau_\psi < \infty) = \sum\limits_{\eta\in\mathscr{G}^*} p(\eta) E_{\theta_0} s_\eta^\psi = 1$, we obtain

$$I(\theta_0,\theta;\psi) \geq -\ln\left(\sum_{\eta\in\mathscr{G}^*} p(\eta) E_\theta s_\eta^\psi\right) \geq 0.$$

Let now (ψ,ϕ) be any group-sequential test such that $P_{\theta_0}(\tau_\psi < \infty) = 1$. Suppose $0 < \beta_{\theta_0}(\psi,\phi) < 1$. Then

$$I(\theta_0,\theta;\psi) = \beta_{\theta_0}(\psi,\phi) \frac{\sum\limits_{\eta\in\mathscr{G}^*} p(\eta) E_{\theta_0} s_\eta^\psi \phi_\eta(-\ln(b_\eta))}{\beta_{\theta_0}(\psi,\phi)} + (1-\beta_{\theta_0}(\psi,\phi)) \frac{\sum\limits_{\eta\in\mathscr{G}^*} p(\eta) E_{\theta_0} s_\eta^\psi (1-\phi_\eta)(-\ln(b_\eta))}{1-\beta_{\theta_0}(\psi,\phi)},$$

where $b_\eta = f_\theta^\eta / f_{\theta_0}^\eta$.

In a manner similar to section 3 in [6], applying Lemma 1 to both fractions on the right-hand side of the latter expression, we get

$$I(\theta_0,\theta;\psi) \geq \beta_{\theta_0}(\psi,\phi) \ln \frac{\beta_{\theta_0}(\psi,\phi)}{\beta_\theta(\psi,\phi)} + (1-\beta_{\theta_0}(\psi,\phi)) \ln \frac{1-\beta_{\theta_0}(\psi,\phi)}{1-\beta_\theta(\psi,\phi)}. \tag{7}$$

If $\beta_{\theta_0}(\psi,\phi) = 0$, then

$$I(\theta_0,\theta;\psi) \geq -\ln(1-\beta_\theta(\psi,\phi)),$$

and if $\beta_{\theta_0}(\psi,\phi) = 1$, then

$$I(\theta_0,\theta;\psi) \geq -\ln \beta_\theta(\psi,\phi)$$

((8), (9) in [6]).

The following lemma (an analog of Wald's identity) is useful, in particular, for estimating the information on the left-hand side of (7).

Lemma 2. *Let $g : \mathfrak{X} \mapsto \mathbb{R}^+$ be a nonnegative measurable function on the space of values of the observed random variable such that $E_\theta g(X) < \infty$, where X is a random variable with the density f_θ, and let $Y_{kj} = g(X_{kj})$. Then for any stopping rule ψ, such that $P_\theta(\tau_\psi < \infty) = 1$,*

$$\sum_{n=1}^{\infty} \sum_{\eta \in \mathscr{G}^n} p(\eta) E_\theta s_\eta^\psi \left(\sum_{k=1}^{n} \sum_{j=1}^{\eta_k} Y_{kj} \right) = E_\theta g(X) \sum_{k=1}^{\infty} E\nu_k P_\theta(\tau_\psi \geq k). \tag{8}$$

Proof. Let, for brevity, $E(\cdot)$ and $P(\cdot)$ denote $E_\theta(\cdot)$ and $P_\theta(\cdot)$, respectively, throughout the proof.

Suppose that the left-hand side of (8) is finite. Then

$$\sum_{n=1}^{\infty} \sum_{\eta \in \mathscr{G}^n} p(\eta) E s_\eta^\psi \left(\sum_{k=1}^{n} \sum_{j=1}^{\eta_k} Y_{kj} \right) = \sum_{n=1}^{\infty} \sum_{k=1}^{n} \sum_{\eta \in \mathscr{G}^n} p(\eta) E s_\eta^\psi \left(\sum_{j=1}^{\eta_k} Y_{kj} \right)$$
$$= \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \sum_{\eta \in \mathscr{G}^n} p(\eta) E s_\eta^\psi \left(\sum_{j=1}^{\eta_k} Y_{kj} \right) \tag{9}$$

(we may change the order of summation, since the source series is finite).

Let $Z_k^{(\eta_k)} = \sum_{j=1}^{\eta_k} Y_{kj}$. It is easy to see that under the assumptions of the lemma

$$\sum_{\eta_k \in \mathscr{G}} P(\nu_k = \eta_k) E Z_k^{(\eta_k)} = \sum_{\eta_k \in \mathscr{G}} P(\nu_k = \eta_k) \eta_k E g(X) < \infty.$$

It is not difficult to see that

$$\sum_{n=k}^{\infty} \sum_{\eta \in \mathscr{G}^n} p(\eta) E s_\eta^\psi Z_k^{(\eta_k)} = \sum_{\eta \in \mathscr{G}^k} p(\eta) E t_\eta^\psi Z_k^{(\eta_k)}.$$

This implies that the right-hand side of (9) equals

$$\sum_{k=1}^{\infty} \sum_{\eta \in \mathscr{G}^k} p(\eta) E t_\eta^\psi Z_k^{(\eta_k)}.$$

By the independence of t_η^ψ and $Z_i^{(\eta_i)}$, $\eta \in \mathscr{G}^k$, we have

$$\sum_{\eta \in \mathscr{G}^k} p(\eta) E t_\eta^\psi Z_k^{(\eta_k)} = \sum_{\eta \in \mathscr{G}^k} p(\eta) E t_\eta^\psi E Z_k^{(\eta_k)} = \sum_{\eta_k \in \mathscr{G}} \eta_k P(\nu_k = \eta_k) E g(X) P(\tau_\psi \geq k),$$

so that

$$\sum_{n=1}^{\infty} \sum_{\eta \in \mathscr{G}^n} p(\eta) E s_\eta^\psi \left(\sum_{k=1}^{n} \sum_{j=1}^{\eta_k} Y_{kj} \right) = E g(X) \sum_{k=1}^{\infty} E\nu_k P(\tau_\psi \geq k).$$

If we start from the assumption that the last sum in the latter expression is finite, we can reverse all the arguments to see that there is an equality in (8). □

Corollary. Suppose that $I(\theta_0, \theta) < \infty$ and $\sup_{k \geq 1} E\nu_k < \infty$. Then for any stopping rule ψ, such that $E_{\theta_0} \tau_\psi < \infty$,

$$I(\theta_0, \theta; \psi) = I(\theta_0, \theta) \sum_{k=1}^{\infty} E\nu_k P_{\theta_0}(\tau_\psi \geq k). \tag{10}$$

Proof repeats the proof of corollary 1 in [6] with

$$Y_{kj} = \ln f_{\theta_0}(X_{kj})/f_\theta(X_{kj})$$

instead of Y_j and reference to Lemma 2 instead of lemma 2 in [6]. □

Since $\sum_{k=1}^{\infty} P(\tau_\psi \geq k) = E\tau_\psi$, under Assumption 1 by Lemma 2 we have that for any $M > \gamma_1$ there exists $\delta > 0$ such that $I(\theta_0, \theta; \psi) \leq (\theta - \theta_0)^2 M E\tau_\psi \sup_{k\geq 1} E\nu_k$, if $|\theta - \theta_0| \leq \delta$.

The following theorem is a consequence of the information inequality (7), it relates the following characteristics of the test: the average number of observations, the type I error, and the derivative of the power function.

Theorem 1. *Suppose that Assumption* 1 (P. 56) *is fulfilled. Then for any group-sequential test* (ψ, ϕ) *such that* $E_{\theta_0}\tau_\psi < \infty$ *and the derivative* $\dot\beta_{\theta_0}(\psi, \phi)$ *of its power function* $\beta_\theta(\psi, \phi)$ *at* $\theta = \theta_0$ *exists, it holds*

$$(\dot\beta_{\theta_0}(\psi, \phi))^2 \leq 2\gamma_1 \beta_{\theta_0}(\psi, \phi)(1 - \beta_{\theta_0}(\psi, \phi)) \sum_{k=1}^{\infty} E\nu_k P_{\theta_0}(\tau_\psi \geq k).$$

Proof is conducted in the same way as the proof of theorem 1 in [6]. □

Remark 1. In case of i. i. d. observations following a distribution from a regular family (i.e., if $\gamma_1 = I(\theta_0)/2$) and i. i. d. group sizes $\nu_1, \nu_2, \ldots$ by Theorem 1 it is easy to see that

$$(\dot\beta_{\theta_0}(\psi, \phi))^2 \leq \beta_{\theta_0}(\psi, \phi)(1 - \beta_{\theta_0}(\psi, \phi)) I(\theta_0) E\nu_1 E_{\theta_0}\tau_\psi$$

([6], remark 1).

The following theorem is an analog of theorem 2 from [6] for group-sequential tests.

Theorem 2. *Let Assumptions* 1–3 (pp. 56, 56) *be fulfilled. Then the power function* $\beta_\theta(\psi, \phi)$ *of any group-sequential test* (ψ, ϕ) *such that* $E_{\theta_0}\tau_\psi < \infty$ *is differentiable at* $\theta = \theta_0$ *and its derivative is*

$$\dot\beta_{\theta_0}(\psi, \phi) = \sum_{\eta \in \mathscr{G}^*} p(\eta) E_{\theta_0}(s_\eta^\psi \phi_\eta z_\eta),$$

where for $\eta \in \mathscr{G}^n$, $n \geq 1$,

$$z_\eta = z_\eta(x_1^{(\eta_1)}, \ldots, x_n^{(\eta_n)}) = \sum_{k=1}^{n} \sum_{j=1}^{\eta_k} \frac{\dot f_{\theta_0}(x_{kj})}{f_{\theta_0}(x_{kj})}$$

(*by definition, we suppose* $\sum_{j=1}^{0}(\cdot) \equiv 0$).

Proof. Let (ψ, ϕ) be any group-sequential test such that $E_{\theta_0}\tau_\psi < \infty$. Let us prove that

$$(\beta_\theta(\psi, \phi) - \beta_{\theta_0}(\psi, \phi))/(\theta - \theta_0) - \sum_{\eta \in \mathscr{G}^*} p(\eta) E_{\theta_0}(s_\eta^\psi \phi_\eta z_\eta) \to 0,$$

$\theta \to \theta_0$, i.e.,

$$\sum_{\eta \in \mathscr{G}^*} p(\eta) \int s_\eta^\psi \phi_\eta ((f_\theta^\eta - f_{\theta_0}^\eta)/(\theta - \theta_0) - \dot f_{\theta_0}^\eta) d\mu^\eta \to 0, \tag{11}$$

$\theta \to \theta_0$, where $\dot f_{\theta_0}^\eta = z_\eta f_{\theta_0}^\eta$ for $\eta \in \mathscr{G}^*$ (it is easy to see that

$$E_{\theta_0} s_\eta^\psi \phi_\eta z_\eta = \int s_\eta^\psi \phi_\eta \dot f_{\theta_0}^\eta d\mu^\eta,$$

since from Assumption 2 it follows that $\dot{f}_{\theta_0} = 0$ μ-almost everywhere on $\{x : f_{\theta_0}(x) = 0\}$).

For any $\eta \in \mathscr{G}^n$, $n \geq 1$, let $|\eta| = \sum_{i=1}^{n} \eta_i$.

Let us first prove

$$\sum_{\eta \in \mathscr{G}^*: |\eta| \leq m} p(\eta) \int s_\eta^\psi \phi_\eta((f_\theta^\eta - f_{\theta_0}^\eta)/(\theta - \theta_0) - \dot{f}_{\theta_0}^\eta) d\mu^\eta \to 0 \tag{12}$$

as $\theta \to \theta_0$ for every fixed $m \geq 1$. Indeed, the absolute value of the left-hand side of (12) is no greater than

$$\sum_{\eta \in \mathscr{G}^*: |\eta| \leq m} p(\eta) \int \left|(f_\theta^\eta - f_{\theta_0}^\eta)/(\theta - \theta_0) - \dot{f}_{\theta_0}^\eta\right| d\mu^\eta$$
$$= \sum_{n=1}^{m} \sum_{\eta \in \mathscr{G}^*: |\eta| = n} p(\eta) \int \left|(f_\theta^\eta - f_{\theta_0}^\eta)/(\theta - \theta_0) - \dot{f}_{\theta_0}^\eta\right| d\mu^\eta. \tag{13}$$

It is easy to get from Assumption 2 that every integral on the right-hand side of (13) tends to zero as $\theta \to \theta_0$ (in essence, this is the differentiability of $f_\theta^{\eta_k} = \prod_{j=1}^{\eta_k} f_\theta$ in $L_1(\mu^{\eta_k})$ under the condition of differentiability of f_θ in $L_1(\mu)$). Thus, we obtain (12).

Therefore, in order to prove (11), it suffices to show that for any $\varepsilon > 0$ there exists $m > 1$ such that

$$\limsup_{\theta \to \theta_0} \left| \sum_{\eta \in \mathscr{G}^*: |\eta| > m} p(\eta) \int s_\eta^\psi \phi_\eta((f_\theta^\eta - f_{\theta_0}^\eta)/(\theta - \theta_0) - \dot{f}_{\theta_0}^\eta) d\mu^\eta \right| < 2\varepsilon. \tag{14}$$

Obviously, (14) will follow if we show that there exists m such that

$$\limsup_{\theta \to \theta_0} \left| \sum_{\eta \in \mathscr{G}^*: |\eta| > m} p(\eta) \int s_\eta^\psi \phi_\eta(f_\theta^\eta - f_{\theta_0}^\eta)/(\theta - \theta_0) d\mu^\eta \right| < \varepsilon, \tag{15}$$

$$\sum_{\eta \in \mathscr{G}^*: |\eta| > m} p(\eta) \int s_\eta^\psi |\dot{f}_{\theta_0}^\eta| d\mu^\eta = \sum_{\eta \in \mathscr{G}^*: |\eta| > m} p(\eta) E_{\theta_0}(s_\eta^\psi |z_\eta|) < \varepsilon. \tag{16}$$

Let us first prove (16). Note that according to Lemma 2

$$\sum_{n=1}^{\infty} \sum_{\eta \in \mathscr{G}^\eta} p(\eta) E_{\theta_0} \left(s_\eta^\psi \sum_{k=1}^{n} \sum_{j=1}^{\eta_k} \left| \frac{\dot{f}_{\theta_0}(X_{kj})}{f_{\theta_0}(X_{kj})} \right| \right) = E_{\theta_0} \left| \frac{\dot{f}_{\theta_0}(X)}{f_{\theta_0}(X)} \right| \sum_{k=1}^{\infty} E\nu_k P_{\theta_0}(\tau_\psi \geq k), \tag{17}$$

where the series on the right-hand side is finite, since Assumptions 2 and 3 are fulfilled.

Hence, the series on the left-hand side of (17) is converging, which implies (16).

Let us now prove that there exists m such that (15) is fulfilled. To this end, let us apply Lemma 1 with $G(x) = -\ln x$, $a_\eta = \phi_\eta I_{\{|\eta| > m\}}$, $b_\eta = f_\theta^\eta / f_{\theta_0}^\eta$, $\eta \in \mathscr{G}^*$.

Let, for brevity,

$$\alpha_m = \sum_{\eta \in \mathscr{G}^*: |\eta| > m} p(\eta) E_{\theta_0} s_\eta^\psi \phi_\eta, \quad \alpha_m(\theta) = \sum_{\eta \in \mathscr{G}^*: |\eta| > m} p(\eta) E_\theta s_\eta^\psi \phi_\eta.$$

Then (15) is equivalent to

$$\limsup_{\theta \to \theta_0} \left| \frac{\alpha_m(\theta) - \alpha_m}{\theta - \theta_0} \right| \leq \varepsilon. \tag{18}$$

Under the assumption that $0 < \alpha_m < 1$, we have

$$I(\theta_0,\theta;\psi)=\alpha_m\frac{\sum\limits_{\eta\in\mathscr{G}^*}p(\eta)E_{\theta_0}s_\eta^\psi\phi_\eta I_{\{|\eta|>m\}}(-\ln(b_\eta))}{\alpha_m}$$
$$+(1-\alpha_m)\frac{\sum\limits_{\eta\in\mathscr{G}^*}p(\eta)E_{\theta_0}s_\eta^\psi(1-\phi_\eta I_{\{|\eta|\geq m\}})(-\ln(b_\eta))}{1-\alpha_m}. \quad (19)$$

Applying Lemma 1 to both fractions on the right-hand side of (19), we obtain

$$I(\theta_0,\theta;\psi)\geq-\alpha_m\ln\frac{\alpha_m(\theta)}{\alpha_m}-(1-\alpha_m)\ln\frac{1-\alpha_m(\theta)}{1-\alpha_m}.$$

The rest of the proof (culminating in (18)) almost literally repeats the respective part of the proof of theorem 2 in [6]. □

Remark 2. Theorem 2 is a generalization, to the case of group-sequential tests and randomized stopping and decision rules, of lemma 4.1.4 in [11]. In remark 2 of [6] there is some history of this type of results.

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The Linear Conjugation Problem for Bi-Analytic Functions

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Abstract—We consider a classical problem on linear conjugation problem for bi-analytic functions on smooth contour. We obtain explicit formula of a solution to a problem and describe necessary and sufficient conditions of its solvability.

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Let an oriented smooth contour Γ consisting of simple curves $\Gamma_1, \dots, \Gamma_m$ be defined on the complex plane. Then its complement $D = \mathbb{C} \setminus \Gamma$ consists of several domains $D_0, D_1, \dots, D_m$ such that the domain D_0 is infinite and contains a neighborhood of the point at infinity ∞, and all other domains are finite. We consider in these domains a bi-analytic function ϕ, i.e., a function $\phi \in C^2(D)$ satisfying the equation

$$\frac{\partial^2 \phi}{\partial \overline{z}^2} = 0.$$

It is well-known [1, 2], that it allows representation in terms of two analytic functions ϕ_0, ϕ_1 by the Goursat formula

$$\phi(z) = \phi_0(z) + \overline{z}\phi_1(z), \quad z \in D, \tag{1}$$

where $\phi_1 = \partial\phi/\partial\overline{z}$.

Let bi-analytic in D function ϕ together with its partial derivative $\phi_1 = \partial\phi/\partial\overline{z}$ be continuous in closed domains $\overline{D}_j$, i.e., it has one-side boundary values $\phi^\pm$ and $\phi_1^\pm$ on Γ. Then there is determined a problem on linear conjugation

$$\phi^+ - G_0\phi^- = f_0, \quad \left(\frac{\partial\phi}{\partial\overline{z}}\right)^+ - G_1\left(\frac{\partial\phi}{\partial\overline{z}}\right)^- = f_1, \tag{2}$$

where coefficients G_k and right-hand parts f_k are given, and $G_k(t) \neq 0$ for any $t \in \Gamma$.

In what follows we assume that the functions G_k and f_k belong to the Hölder class $C^\mu(\Gamma)$, and solutions belong to the class

$$\phi, \phi_1 \in C^\mu(\overline{D}_j), \ 1 \le j \le m; \quad \phi, \phi_1 \in C^\mu(\overline{D}_0 \cap \{|z| \le R\}), \tag{3}$$

where $\phi_1 = \partial\phi/\partial\overline{z}$, and $R > 0$ is selected so that $\Gamma \subseteq \{|z| < R\}$. Additionally, for certain integer k the behavior of ϕ at infinity point satisfies the bound

$$|\phi(z)| + |z|\,|\phi_1(z)| \le C|z|^{k-1} \ \text{for} \ |z| \ge R. \tag{4}$$

The problems of similar type are studied by a number of authors (e.g., [3, 4]), as a rule, for functions bounded at the infinity point. The scheme of its solving is well-known [5]. It reduces to certain problems of analogous type for analytic functions by means of representation (1). But in general case of arbitrary

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functions G_0, G_1 the explicit solution was not obtained (e.g., [5], pp. 318, 319). In the present paper we give the explicit solution of the problem for any k in bound (4) and describe exact conditions for its solvability.

Let $\varkappa_k = \operatorname{Ind} G_k$, $k = 0, 1$, be the Cauchy index of function G_k, i.e., if there are fixed points $\tau_j \in \Gamma_j$ and continuous on $\Gamma_j \setminus \tau_j$ branches of $\arg G_k$, then

$$\varkappa_k = \frac{1}{2\pi} \sum_{j=1}^{m} [(\arg G_k)(\tau_j - 0) - (\arg G_k)(\tau_j + 0)], \tag{5}$$

where one-side limit values at the points τ_j are understood relatively the orientation on the curve Γ_j. Let X_k be canonical function of the problem of linear conjugation corresponding to coefficient G_k. Let us recall [6] that this function does not vanish everywhere including its limit values $X_k^{\pm}$, satisfies the boundary-value condition

$$X_k^{+} = G_k X_k^{-} \tag{6}$$

and the restriction

$$\lim_{z \to \infty} z^{\varkappa_k} X_k(z) = 1 \tag{7}$$

at the infinity. As known [6], the function X_k with these properties is determined uniquely and $X_k^{\pm} \in C^{\mu}(\Gamma)$.

We put

$$A(t) = \frac{\overline{t}[G_1(t) + G_0(t)]}{2G_1(t)}, \quad B(t) = \frac{\overline{t}[G_1(t) - G_0(t)]}{2G_1(t)}, \quad C(t) = \frac{B(t) X_1^{+}(t)}{X_0^{+}(t)}, \tag{8}$$

and introduce singular operator

$$(N\varphi)(t_0) = A(t_0) + \varphi(t_0) + \frac{B(t_0)}{\pi i} \int_{\Gamma} \frac{X_1^{+}(t_0)}{X_1^{+}(t)} \frac{\varphi(t) dt}{t - t_0}, \quad t_0 \in \Gamma. \tag{9}$$

We denote by $P(n)$ the class of polynomials $p(t)$ of degree $\deg p \leq n - 1$, where n is integer; for $n \leq 0$ we put $P(n) = 0$. Thus, $\dim P(n) = \max(0, n)$. If m is integer, then it is convenient to introduce subspace $P(n, m)$ of all polynomials $p \in P(n)$ such that

$$\langle q, Cp \rangle = 0, \quad q \in P(m), \tag{10}$$

where $\langle \varphi, \psi \rangle = \int\limits_{\Gamma} \varphi(t) \psi(t) dt$.

This subspace arises in the following situation.

Lemma. *Let a function $f \in C(\Gamma)$ satisfies the condition*

$$\langle f, p \rangle = \langle q, Cp \rangle, \quad p \in P(n), \tag{11}$$

for certain polynomial $q \in P(m)$. Then the condition is equivalent to

$$\langle f, p \rangle = 0, \quad p \in P(n, m).$$

Proof. We consider without loss of generality that values m and n are positive. Expand polynomials p and q in basis functions $e_i(t) = t^{i-1}$, $i = 1, 2, \ldots$, in explicit form

$$p = \sum_{i=1}^{n} x_i e_i, \quad q = \sum_{j=1}^{m} y_j e_j,$$

with certain $x_i, y_j \in \mathbb{C}$. Then we are able to rewrite (11) as the identity

$$\sum_{i=1}^{n} x_i \langle f, e_i \rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} x_i y_j \langle e_j, Ce_i \rangle$$

for $x \in \mathbb{C}^n$, what is equivalent to solvability of system of linear equations

$$\sum_{j=1}^{m} \langle e_j, Ce_i \rangle y_j = \langle f, e_i \rangle, \quad 1 \le i \le n.$$

Clearly, the system is solvable if and only if its right part satisfies the orthogonality condition

$$\sum_{i=1}^{n} \langle f, e_i \rangle \xi_i = 0 \tag{12}$$

to all solutions $\xi = (\xi_1, \dots, \xi_n)$ of dual homogeneous system

$$\sum_{i=1}^{n} \langle e_j, Ce_i \rangle \xi_i = 0, \quad 1 \le j \le m. \tag{13}$$

Putting $p = \sum_1^n \xi_i e_i$, we rewrite equality (12) in the form $\langle f, p \rangle = 0$ for all polynomials $p \in P(n)$ satisfying condition $\langle e_j, Cp \rangle = 0$, $1 \le j \le m$, or, what is equivalent, condition (10).

A reader concludes from the proof of Lemma that dimension of the subspace $P(n,m)$ coincides with the number of linearly independent solutions to homogeneous system (13). Thus,

$$\dim P(n,m) = n - \operatorname{rang} C(n,m), \tag{14}$$

where the matrix $C(n,m) \in \mathbb{C}^{m\times n}$ is determined by elements $C_{ij} = \langle e_j, Ce_i \rangle$ and has the following structure:

$$C(n,m) = \begin{pmatrix} c_1 & c_2 & \cdots & c_n \\ c_2 & c_3 & \cdots & c_{n+1} \\ \dots & \dots & \dots & \dots \\ c_m & c_{m+1} & \cdots & c_{m+n-1} \end{pmatrix}, \quad c_k = \int_\Gamma C(t) t^{k-1} dt.$$

Let us formulate the main result on solvability of problem (2)–(4).

Theorem. *Problem* (2) *is solvable in the class* (3), (4) *if and only if its right-hand parts* f_0, f_1 *satisfy orthogonality conditions*

$$\begin{gathered} \langle f_1, (X_1^+)^{-1} q_1 \rangle = 0, \quad q_1 \in P(-\varkappa_1 - k + 1), \\ \langle f_0 - N f_1, (X_0^+)^{-1} q_0 \rangle = 0, \quad q_0 \in P(-\varkappa_0 - k, \varkappa_1 + k - 1). \end{gathered} \tag{15}$$

All solutions of the problem under these conditions are described by the formula

$$\phi(z) = \frac{1}{2\pi i} \int_\Gamma \left[\frac{X_0(z)}{X_0^+(t)} \frac{f_0(t) - (N f_1)(t)}{t - z} + \frac{\overline{z} X_1(z)}{X_1^+(t)} \frac{f_1(t)}{t - z} \right] dt \\ - \frac{1}{\pi i} \int_\Gamma \left[\frac{X_0(z)}{X_0^+(t)} \frac{B(t) X_1^+(t) p_1(t)}{t - z} \right] dt + X_0(z) p_0(z) + \overline{z} X_1(z) p_1(z) \tag{16}$$

with arbitrary $p_0 \in P(\varkappa_0 + k)$ *and* $p_1 \in P(\varkappa_1 + k - 1)$.

Proof. According to (1), problem (2) is reducible to equivalent system of two problems for two analytical functions:

$$\phi_1^+ - G_1 \phi_1^- = f_1, \quad \phi_0^+ - G_0 \phi_0^- = f_2, \tag{17}$$

where $f_2(t) = f_0(t) - \overline{t}[\phi_1^+(t) - G_0(t)\phi_1(t)]$. These problems are considered in class of functions (3) with corresponding bounds

$$|\phi_1(z)| \le C|z|^{k-2}, \tag{18_1}$$

$$|\phi_0(z)| \le C|z|^{k-1}, \tag{18_0}$$

near the point at infinity, which follow from (4).

The known general results [6, 7] on the problem of linear conjugation imply by virtue of (6), (7) that the first problem for ϕ_1 is solvable in the class (3), (18_1) if and only if

$$\langle f_1, (X_1^+)^{-1}q_1\rangle = 0, \quad q_1 \in P(-\varkappa_1 - k + 1), \tag{18}$$

and under these conditions all its solutions are given by the formula

$$\phi_1(z) = \frac{1}{2\pi i}\int_\Gamma \frac{X_1(z)}{X_1^+(t)}\frac{f_1(t)dt}{t-z} + X_1(z)p_1(z), \quad p_1 \in P(\varkappa_1 + k - 1). \tag{19}$$

We evaluate function f_2 by means of this formula, and write it in the form

$$f_2(t) = f_0(t) - \bar{t}f_1(t) - \bar{t}[G_1(t) - G_0(t)]\phi_1^-(t). \tag{20}$$

By means of (6) and the Sokhotskii–Plemelj formula, applied to the Cauchy type integral in (20), we have

$$2G_1(t_0)\phi_1^-(t_0) = -f_1(t_0) + \frac{1}{\pi i}\int_\Gamma \frac{X_1^+(t_0)}{X_1^+(t)}\frac{f_1(t)dt}{t-t_0} + 2X_1^+(t_0)p_1(t_0).$$

We substitute this representation into (21), and obtain in notation (8), (9)

$$f_2(t) = f_0(t) - (Nf_1)(t) - 2B(t)X_1^+(t)p_1(t). \tag{21}$$

As above, the second problem in (17) is solvable in the class (3), (18_0) if and only if

$$\langle f_2, (X_0^+)^{-1}q_0\rangle = 0, \quad q_0 \in P(-\varkappa_0 - k), \tag{22}$$

and under these conditions all its solutions are given by the formula

$$\phi_0(z) = \frac{1}{2\pi i}\int_\Gamma \frac{X_0(z)}{X_0^+(t)}\frac{f_2(t)dt}{t-z} + X_0(z)p_0(z), \quad p_0 \in P(\varkappa_0 + k). \tag{23}$$

Let us consider in detail condition (23). According to (8), (18) we rewrite it as the identity

$$\langle f_0 - Nf_1, (X_0^+)^{-1}q_0\rangle = 2\langle p_1, Cq_0\rangle, \quad q_0 \in P(-\varkappa_0 - k), \tag{24}$$

for certain polynomial $p_1 \in P(\varkappa_1 + k - 1)$. By virtue of Lemma, where we have to substitute function $(2X_0^+)^{-1}(f_0 - Nf_1)$ instead of f and swap the letters p, q, condition (25) is equivalent to the second orthogonality condition (15), and, consequently, we can replace conditions (19), (25) by (15). The substitution of (20) and (22), (24) into (1) implies formula (16). This concludes the proof.

The theorem means that the number of linearly independent solutions to homogeneous problem is $\dim P(\varkappa_0 + k) + \dim P(\varkappa_1 + k - 1)$, and the number of linearly independent conditions for its solvability is $\dim P(-\varkappa_0 - k, \varkappa_1 + k - 1) + \dim P(-\varkappa_1 - k + 1)$. Hence, the index $\varkappa$ of the problem is given by the formula

$$\varkappa = \dim P(\varkappa_0 + k) + \dim P(\varkappa_1 + k - 1) - \dim P(-\varkappa_0 - k, \varkappa_1 + k - 1) - \dim P(-\varkappa_1 - k + 1).$$

By means of (14) we obtain

$$\varkappa = \varkappa_0 + \varkappa_1 + 2k - 1 + s, \quad s = \mathrm{rang} C(-\varkappa_0 - k, \varkappa_1 + k - 1).$$

If one of numbers $-\varkappa_0 - k, \varkappa_1 + k - 1$ is negative, then $s = 0$ in this equality.

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Summability of Fourier Series for Almost-Periodic Functions on Locally Compact Abelian Groups

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Abstract—Some results concerning the summability of Fourier series of continuous 2π-periodic functions are generalized for the case of almost-periodic functions defined on locally compact Abelian groups.

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INTRODUCTION

The theory of almost periodic functions, created by Bohr in the twenties of the last century, encouraged the development of harmonic analysis on locally compact groups. We can associate with every almost periodic function, defined on a locally compact group, its Fourier series, and the problem of its summability naturally arises.

In the paper we investigate the problem of summability of Fourier series of almost-periodic functions defined on locally compact Abelian groups. We obtain analogs of well-known classical results described in monograph [1]. The main our tool is the use of convolution operators investigated in [2–4]. Kernels of these operators are products of Fourier transforms of characteristic functions for the closures of symmetric neighborhoods of the unit in the dual group. With the help of these operators, in [2–4] some analogs of entire functions of exponential type for locally compact Abelian groups were constructed and the problem of summability of Fourier integrals for some classes of functions was studied. In Theorem 1 we solve the problem of summability of Fourier transforms for the case when the characters, appearing in the Fourier expansion, have a unique limiting point at the infinity of the dual space. The case when a unique limiting point is the unit character is investigated in Theorem 2. Moreover, we introduce various analogs of the Fejer–Bochner means for Fourier series and study their approximative properties (Theorems 3 and 4).

1. MAIN CONCEPTS AND FACTS

Consider a locally compact Abelian group G with Haar measure μ, assuming that the topology of G is Hausdorff. We denote by $\widehat{G}$ the group dual to G, i.e., the group of all continuous homomorphisms (characters) acting from G to the group which is the unit circle endowed with the topology of uniform convergence on compact subsets of G. We denote by $L^1(G)$ the space of all functions integrable on G by μ. For a functions $f \in L^1(G)$ we can define its Fourier transform $\widehat{f}$ which is defined on $\widehat{G}$ according to the equality $\widehat{f}(\chi) = \int\limits_G f(g)\overline{\chi(g)}d_\mu(g)$, $\chi \in \widehat{G}$. If ν is the Haar measure on $\widehat{G}$, then for a function $F \in L^1(\widehat{G})$ we can define the inverse Fourier transform $\widetilde{F}$ by the equality $\widetilde{F}(g) = \int\limits_{\widehat{G}} F(\chi)\chi(g)d_\nu(\chi)$,

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$g \in G$. Below we will simply write dh instead of $d_\mu h$. In the future we will assume that the Haar measures on G and $\widehat{G}$ are normalized such that the inversion formula $f = (\widehat{f})^\sim$ is valid if $f \in L^1(G)$ and $\widehat{f} \in L^1(\widehat{G})$.

Let U_G be the set of all symmetric compact sets in G which are the closures of neighborhoods of the unit in G. For arbitrary T and K from $U_{\widehat{G}}$ we consider the product $TK = \{g_1 g_2 : g_1 \in T, g_2 \in K\}$ and the function

$$V_{T,K}(g) = (\operatorname{mes} T)^{-1}\widehat{(1)}_T(g)\widehat{(1)}_{TK}(g), \ \ g \in G, \tag{1}$$

where, for example, $\widehat{(1)}_T$ means the Fourier transform of the characteristic function of T. In [2] some approximative properties of the convolution

$$f_{T,K}(g) = (f * V_{T,K})(g) = \int_G f(h^{-1}g)V_{T,K}(h)d_\mu h, \ \ g \in G, \tag{2}$$

are investigated for some classes of functions f, defined on G.

In the limiting case, when K is the unit element e of $\widehat{G}$, (1) takes the form $V_T := V_{T,e} = (\operatorname{mes} T)^{-1}(\widehat{(1)_T(g)})^2$; it is an analog of the known Fejer kernel. The convolution operator $(f * V_T)(g)$ corresponding to the kernel is considered in [3, 4].

Let

$$\varphi_{T,K}(\chi) = (\operatorname{mes} T)^{-1}((1)_T * (1)_{TK})(\chi) = (\operatorname{mes} T)^{-1}\int_T (1)_{TK}(\chi h)dh, \ \ \chi \in \widehat{G}. \tag{3}$$

Calculating the Fourier transform of the function by the known formula [5], we obtain that $V_{T,K}(g) = (\operatorname{mes} T)^{-1}\widehat{\varphi_{T,K}}(g)$. If $\chi \in K$, then $\chi h \in TK$, $(1)_{TK}(\chi h) = 1$, and $\varphi_{T,K}(\chi) = 1$. If $\chi \notin KT^2$, then $\chi h \notin KT$ for all $h \in T$ and $\varphi_{T,K}(\chi) = 0$. It is clear that $|\varphi_{T,K}(\chi)| \leq 1$ for all $\chi \in \widehat{G}$.

We note that if $\operatorname{mes} T > 0$, $T, K \in U_{\widehat{G}}$, then $V_{T,K}$ belongs to $L^1(G)$. Indeed, using the Cauchy–Bunyakovsky–Schwarz inequality and the Plancherel theorem [5], we obtain

$$\begin{aligned}\|V_{T,K}\|_1 &\leq (\operatorname{mes} T)^{-1}\|\widehat{(1)}_T\|_2\|\widehat{(1)}_{TK}\|_2 \\ &= (\operatorname{mes} T)^{-1}\|(1)_T\|_2\|(1)_{TK}\|_2 = ((\operatorname{mes} T)^{-1}\operatorname{mes}(TK))^{1/2}. \end{aligned} \tag{4}$$

Here and below we denote by $\|\cdot\|_1$ and $\|\cdot\|_2$ the usual norm in $L^1(G)$ or $L^1(\widehat{G})$ and the norm of $L^2(G)$ or $L^2(\widehat{G})$. We note that

$$\begin{aligned}\int_G V_{T,K}(g)dg &= (\operatorname{mes} T)^{-1}\int_G \widehat{(1)}_T(g)\widehat{(1)}_{KT}(g)dg \\ &= (\operatorname{mes} T)^{-1}\int_{\widehat{G}} (1)_T(g)(1)_{KT}(g)dg = (\operatorname{mes} T)^{-1}\int_T dg = 1.\end{aligned}$$

A complex-valued function f, defined on G, is called almost-periodic (by Bochner) if the family of function of the form $f(gh)$ (h passes along the whole group G) is compact in the topology of uniform convergence on the whole group, i.e., every infinite sequence $f(gh_k)$, $k \in \mathbb{N}$ has a convergent subsequence. This definition can be introduced for arbitrary topological groups G ([1], P. 255; [6]). For functions defined on locally compact groups there is the concept of almost periodicity in the sense of Bohr [7]. Namely, for $\varepsilon > 0$ an element $w \in G$ is called a ε-translation of a function f, defined on a locally compact group G, if the condition $|f(xy) - f(xwy)| < \varepsilon$ holds for all $x, y \in G$. A function f is called almost periodic by Bohr if for every $\varepsilon > 0$ there exists a compact set C in G such that for every element $z \in C$ the set zC contains some ε-translation of f. Struble [7] proved that for complex-valued continuous functions on a locally compact group G the above definitions of almost periodicity are equivalent. We should note that every continuous on a locally compact group almost periodic function are uniformly continuous on it. Other equivalent definitions of almost periodicity on locally compact groups are given in [8].

Below we will denote by $Q(G)$ the space of all almost continuous periodic functions on a locally compact Abelian group G.

Consider $f \in Q(G)$ and its Fourier series (see, e.g., [6])

$$f(g) \sim \sum_{n=0}^{\infty} A_n(f)\chi_n(g), \;\; g \in G, \;\; \chi_n \in \widehat{G}, \;\; \chi_0 = e, \tag{5}$$

where the set of the characters χ_n depends on f. It is clear that $\chi_n \in Q(G)$, therefore, $f_{T,K}\chi_n \in Q(G)$, $n \in \mathbb{N}$. The Fourier coefficient $A_n(f) = M_g\{f(g)\overline{\chi_n(g)}\}$ is the mean value of the function $f(g)\overline{\chi_n(g)}$ almost periodic on G. In [7] it is proved that if G is a locally compact group, μ is its left Haar measure, and $\mathcal{A} = [U_t]$ is a so-called left sampler family of subsets in G, then for every measurable almost periodic function f on G its mean value $M_g\{f(g)\}$ coincides with $\lim\limits_{t\to\infty}(\mu(U_t))^{-1}\int_{U_t} f(x)d\mu(x)$.

2. MAIN RESULTS

At first, we consider the problem of summability of Fourier series in the case when every set $K \in U_{\widehat{G}}$ contains only a finite number of the characters χ_n from (5). In this case elements of the form $\sum\limits_{\chi_k\in K} a_k\chi_k(g)$, $K \in U_{\widehat{G}}$, $g \in G$, with complex coefficients $a_k \in \mathbb{C}$ produce a finite-dimensional subspace in $Q(G)$; we denote it by $P_K(f)$. Its elements we will call trigonometric polynomials of degree K, generated by $f \in Q(G)$, or simply trigonometric polynomials of degree K.

Lemma 1. *Let the Fourier series of $f \in Q(G)$ have the form (5) and every set from $U_{\widehat{G}}$ contain only a finite number of the characters χ_n from* (5). *If $T, K \in U_{\widehat{G}}$, then*

$$f_{T,K}(g) = \sum_{\chi_k\in KT^2} A_k(f)\varphi_{T,K}(\chi_k)\chi_k(g). \tag{6}$$

where $V_{T,K}$, $f_{T,K}$, and $\varphi_{T,K}$ are defined by (1)–(3),

Proof. Due to symmetry of T and K, in the right-hand side of (2) we can replace $f(h^{-1}g)$ with $f(hg)$. From (2) and (4) it follows that for every $\tau \in G$

$$\begin{aligned}|f_{T,K}(g\tau) - f_{T,K}(g)| &\le \int_G |f(gh\tau) - f(gh)| \cdot |V_{T,K}(h)|dh \\ &\le \sup_{g\in G}|f(g\tau) - f(g)| \cdot ((\operatorname{mes} T)^{-1}\operatorname{mes}(TK))^{1/2}.\end{aligned}$$

It means that $f_{T,K} \in Q(G)$. The Fourier coefficient B_k of $f_{T,K}$, corresponding to the character χ_k, has the form

$$\begin{aligned}B_k = M_g\{f_{T,K}\cdot\chi_k\} &= M_g\Big\{\int_G f(gh)V_{T,K}(h)dh \cdot \chi_k(g)\Big\} \\ &= M_g\Big\{\int_G V_{T,K}(h)\chi_k(h^{-1})f(gh)\chi_k(gh)dh\Big\}.\end{aligned}$$

For every function $f \in Q(G)$ its mean value can be approximated, uniformly with respect to $c \in G$, by expressions of the form $n^{-1}\sum\limits_{i=1}^{n} f(cg_i)$; here $g_i \in G$, $i = 1, \ldots, n$, depend on n ([1], P. 260). Therefore,

$$B_k = \lim_{n\to\infty} n^{-1}\sum_{i=1}^{n}\int_G f(cg_ih)\chi_k(cg_ih)V_{T,K}(h)\chi_k(h^{-1})dh.$$

Taking into account (4) and the fact that every almost periodic function is bounded on G, we can pass to the limit under the integral. After application of known properties of mean value (see [6] or [1], P. 262) we obtain

$$B_k = \int_G V_{T,K}(h)\chi_k(h^{-1})M_g\{f(gh)\chi_k(gh)\}dh$$
$$= \int_G V_{T,K}(h)\chi_k(h^{-1})M_g\{f(g)\chi_k(g)\}dh$$
$$= \widehat{V_{T,K}}(\chi_k)M_g\{f(g)\chi_k(g)\} = A_k\varphi_{T,K}(\chi_k). \quad (7)$$

If $\chi_k \notin KT^2$, then from the above properties of $\varphi_{T,K}$ it follows that $B_k = 0$. From this and (7) we conclude that the right-hand side of (6) is the Fourier series of $f_{T,K}(g)$. By assumptions of Lemma 1, the right-hand side of (6) is a finite sum for every T and K, and by the uniqueness theorem for Fourier series (see, e.g., [1], P. 276) we have (6). □

In the case $G = \mathbb{R}$, every character χ_n has the form $\chi_n(x) = e^{i\lambda_n x}$, $x \in \mathbb{R}$, $\lambda_n \in \mathbb{R}$, and if K and T are symmetric neighborhoods of the origin in $\mathbb{R}$, the statement of Lemma 1 is known (see [1], pp. 76, 91, or [9]).

Lemma 2. *If $T, K, U \in U_{\widehat{G}}$, $K \subset U$, and $f(g)$, $g \in G$, is a trigonometric polynomial of degree K, then $f(g) = \int\limits_G f(gh)V_{T,U}(h)dh$, $g \in G$.*

Proof. Let $f(g) = \sum\limits_{\chi_k \in K} A_k(f)\chi_k(g)$ be a trigonometric polynomial of degree K. If $\chi_k \notin K$, then the Fourier coefficient A_k equals zero and the polynomial has the form $f(g) = \sum\limits_{\chi_k \in U} A_k(f)\chi_k(g)$. Now, taking into account that $\varphi_{T,U}(\chi) = 1$ for $\chi \in U$, we see that Lemma 2 follows from (6). □

Let the Fourier series of a function $f \in Q(G)$ have the form (5), $K \in \widehat{G}$ and $E_K(f) = \inf\limits_{a_k \in \mathbb{C}} \sup\limits_{g \in G} |f(g) - \sum\limits_{\chi_k \in K} a_k\chi_k(g)|$ be the best approximation of $f \in Q(G)$ in the subspace $P_K(f)$.

Lemma 3. *If $T, K \in U_{\widehat{G}}$, $f \in Q(G)$, and $f_{T,K}$ is defined by (2), then*

$$|f(g) - f_{T,K}(g)| \le (1 + ((\operatorname{mes} T)^{-1} \operatorname{mes}(TK))^{1/2})E_K(f), \quad g \in G. \quad (8)$$

Proof. Assume that $f_K(g)$ is the polynomial of the best approximation of degree K for $f \in Q(G)$ in $P_K(f)$. From the above two lemmas and (4) it follows that

$$|f(g) - f_{T,K}(g)| \le |f(g) - f_K(g)| + |P_{T,K}(f - f_K)(g)|$$
$$\le E_K(f) + \sup_{g\in G}|f(g) - f_K(g)|\,\|V_{T,K}\|_1 \le (1 + ((\operatorname{mes} T)^{-1} \operatorname{mes}(TK))^{1/2})E_K(f),$$

and (8) is proved. □

Theorem 1. *Let I be an ordered unbounded subset in $\mathbb{R}^+$, and sets $T_\alpha, K_\alpha \in U_{\widehat{G}}$, $\alpha \in I$, satisfy the following conditions:*

a) *if $\alpha < \beta$, $\alpha, \beta \in I$, then $T_\alpha \subset T_\beta$, $K_\alpha \subset K_\beta$;*

b) *the numbers $(\operatorname{mes} T_\alpha)^{-1} \operatorname{mes}(T_\alpha K_\alpha)$ are bounded uniformly by $\alpha \in I$;*

c) *for every fixed set $U \in U_{\widehat{G}}$ the equality $\lim\limits_{\alpha\to\infty} (\operatorname{mes} T_\alpha)^{-1} \operatorname{mes}(T_\alpha U) = 1$ is valid.*

Then for every $f \in Q(G)$, satisfying the assumptions of Lemma 1,

$$f_{T_\alpha,K_\alpha}(g) \to f(g), \quad (9)$$

if $\alpha \to \infty$. Moreover, the convergence is uniform on the whole group G.

Proof. Consider the case when there exist a set $K' \in U_{\widehat{G}} \in U_{\widehat{G}}$ such that $K_\alpha \subset K'$ for every $\alpha \in I$. First we will prove (9) for arbitrary trigonometric polynomial $P \in P_U(f)$ of degree $U \in U_{\widehat{G}}$. If $U \subset K_{\alpha_0}$ for some $\alpha_0 \in I$, then, by Lemma 2, $P(g) - P_{T_\alpha,K_\alpha}(g) = 0$ for all $\alpha \geq \alpha_0$ and (9) is valid. If there is no such α_0, then we consider some set $K^* \in U_{\widehat{G}}$ containing all K_α and U. According to Lemma 2, $P(g) = P_{T_\alpha,K^*}(g)$ and

$$P(g) - P_{T_\alpha,K_\alpha}(g) = (\operatorname{mes} T_\alpha)^{-1} \int_G P(gh)\widehat{(1)}_{T_\alpha}(h)[\widehat{(1)}_{T_\alpha K^*}(h) - \widehat{(1)}_{T_\alpha K_\alpha}(h)]dh.$$

Applying the Cauchy–Bunyakovsky–Schwarz inequality and the Plancherel theorem we obtain

$$\begin{aligned} |P(g) - P_{T_\alpha,K_\alpha}(g)| &\leq (\operatorname{mes} T_\alpha)^{-1}\|P(gh)\widehat{(1)}_{T_\alpha}(h)\|_2\|\widehat{(1)}_{T_\alpha K^*}(h) - \widehat{(1)}_{T_\alpha K_\alpha}(h)\|_2 \\ &\leq (\operatorname{mes} T_\alpha)^{-1} \sup_{g\in G}|P(g)|\, \|(1)_{T_\alpha}(\chi)\|_2\|(1)_{T_\alpha K^*}(\chi) - (1)_{T_\alpha K_\alpha}(\chi)\|_2 \\ &= (\operatorname{mes} T_\alpha)^{-1/2} \sup_{g\in G}|P(g)| \operatorname{mes}(T_\alpha K^* \setminus T_\alpha K_\alpha)^{1/2} \\ &= \sup_{g\in G}|P(g)|(\operatorname{mes}(T_\alpha K^*)/\operatorname{mes} T_\alpha - \operatorname{mes}(T_\alpha K_\alpha)/(\operatorname{mes} T_\alpha)^{1/2}. \end{aligned}$$

By condition c), the expression in the right-hand side tends to zero as $\alpha \to \infty$, therefore, (9) is valid for trigonometric polynomials. Next, according to (4) and condition b), we have

$$\sup_{g\in G}|f_{T_\alpha,K_\alpha}(g)| \leq \sup_{g\in G}|f(g)|\, \|V_{T_\alpha,K_\alpha}\|_1 \leq \sup_{g\in G}|f(g)|(\operatorname{mes}(T_\alpha)^{-1}\operatorname{mes}(T_\alpha K_\alpha)^{1/2})^{1/2} \leq C_1$$

where C_1 does not depend on $\alpha \in I$. The set of trigonometric polynomials is dense in $Q(G)$ by the approximation theorem (see [6] or [1], P. 282). More precisely, $f \in Q(G)$ can be approximated uniformly by polynomials from $\{P_U(f),\, U \in U_{\widehat{G}}\}$. Therefore, in the case $K_\alpha \subset K'$, $\alpha \in I$, Theorem 1 follows from the Banach–Steinhaus theorem. If it is not the case, Theorem 1 follows from Lemma 3, because $\lim\limits_{\alpha\to\infty} E_{K_\alpha}(f) = 0$ according to the approximation theorem mentioned above. □

Remark. From the proof of Theorem 1 we see that uniform convergence in (9) takes place if we do not impose restrictions on the set of characters appearing in the Fourier expansion (5) of $f \in Q(G)$, but conditions a), b), and c) hold and the functions f_{T_α,R_α} are defined by (2).

By (2), (3), and Lemma 1, if $K_\alpha \equiv e$, $\alpha \in I$, then functions (2) take the form

$$f_{T_\alpha}(g) = (\operatorname{mes} T_\alpha)^{-1} \int_G f(gh)(\widehat{(1)}_{T_\alpha}(h))^2 dh = (\operatorname{mes} T_\alpha)^{-1} \sum_{\chi_k \in T_\alpha^2} A_k \int_{T_\alpha} (1)_{T_\alpha}(\chi_k h)dh.$$

We can say that the functions are means of Fejer type for the Fourier series (5) of f defined on G. According to Theorem 1, in the case we obtain that $f_{T_\alpha}(g) \to f(g)$ uniformly on G, if $T_\alpha \to \widehat{G}$ and $\lim\limits_{\alpha\to\infty} \operatorname{mes}(T_\alpha U)/\operatorname{mes} T_\alpha = 1$ for arbitrary fixed set $U \in U_{\widehat{G}}$. It is a generalization of the result from [3] and [4] for the case of almost periodic functions. Consider the case when G is the additive group of real numbers $\mathbb{R}$, $I = \mathbb{R}^+$, $a, b \in I$, $T = (a-b, b-a)$, $K = (-a, a)$, $0 < a < b$, and a/b is less than a fixed number $\theta < 1$. Then the group $\widehat{G}$ is homeomorphic to $\mathbb{R}$, the character corresponding to $x \in G$ has the form $\chi_x(t) = e^{itx}$, $t \in \mathbb{R}$, and $\widehat{(1)}_T(x) = (2\pi)^{-1/2} \int\limits_T e^{-itx}dt = \sqrt{2}\sin ax/(\sqrt{\pi}x)$. Thus, the functions $f_{T,K} := f_{a,b}$, defined by (2), have the form $f_{a,b}(x) = \int\limits_{-\infty}^{\infty} f(x+u)\psi_{a,b}(u)du$ where $\psi_{a,b} =$

$2\pi^{-1}(b-a)^{-1}x^{-2}\sin\frac{b-a}{2}x\sin\frac{b+a}{2}x$ is the Fourier transform of the function

$$\varphi_{a,b}(x)=\begin{cases}1, & |x|\le a;\\ (b-a)^{-1}(b-|x|), & a<|x|<b;\\ 0, & |x|\ge b.\end{cases}$$

For this case the statement of Theorem 1 coincides with the well-known result ([1], Chap. 1, P. 77) that $f_{a,b}(x)\to f(x)$ as $b\to\infty$ uniformly on $\mathbb{R}$ for every continuous almost periodic on $\mathbb{R}$ function f.

For illustration of Theorem 1 we give one more example. Let p be a prime number and $G=Q_p$ be the field of p-adic numbers. With respect to the addition of p-adic numbers, Q_p is a locally compact Abelian group [10]. Its dual group is isomorphic to the additive group Q_p [10]. The character χ_ξ, corresponding to a p-adic number ξ, has the form $\chi_\xi(x)=\exp(2\pi i\{\xi x\}_p)$ [10]. Here the symbol $\{x\}_p$ is defined through p-adic expansion of $x=\sum\limits_{n\ge \operatorname{ord}_p(x)} a_np^n$ by the equality

$$\{x\}_p=\begin{cases}\sum\limits_{n=\operatorname{ord}_p(x)}^{-1} a_np^n, & \text{if } \operatorname{ord}_p(x)<0;\\ 0, & \text{if } \operatorname{ord}_p(x)\ge 0,\end{cases}$$

and ξx is the product of p-adic numbers ξ and x in the field $G=Q_p$.

Let $n_\alpha, m_\alpha\in Z$, T_α and K_α be p-adic balls of radii p^{n_α} and p^{m_α} with centers at zero, i.e., for example, $T_\alpha=\{h\in Q_p: |h|_p\le p^{n_\alpha}\}$ where $|h|_p$ is the p-adic norm of h. The Haar measure of the ball is $\operatorname{mes}T_\alpha=p^{n_\alpha}$ [10]. With the help of the equality

$$\int_{T_\alpha}\chi_\xi(x)dx=\begin{cases}p^{n_\alpha}, & \text{if } |\xi|_p\le p^{-n_\alpha};\\ 0, & \text{if } |\xi|_p\ge p^{-n_\alpha+1},\end{cases}$$

(see [10]) we obtain that in our case

$$f_{T_\alpha,K_\alpha}(x)=p^{-n_\alpha}\int_{Q_p}f(\xi+x)d\xi\int_{|t|_p\le p^{n_\alpha}}\exp(2\pi i\{t\xi\}_p)dt$$
$$\cdot\int_{|t|_p\le p^{n_\alpha+m_\alpha}}\exp(2\pi i\{t\xi\}_p)dt=p^{-n_\alpha-m_\alpha}\int_{|\xi|_p\le p^{-n_\alpha-m_\alpha}}f(\xi+x)d\xi,\ \ x\in Q_p.$$

If f is an arbitrary function from $Q(Q_p)$, then, under assumptions of Theorem 1, we have $\lim\limits_{\alpha\to\infty}f_{T_\alpha,K_\alpha}(x)=f(x)$ uniformly on the whole additive group Q_p.

If in the latter two examples the characters of f have a unique limiting point at the infinity, then the considered approximating aggregates are finite means of its Fourier series, i.e., trigonometric polynomials generated by f.

Now we will investigate the case when a unique limiting point of the characters appearing in the Fourier expansion (5) is a character χ_0 distinct from the infinite point of the group $\widehat{G}$. Without loss of generality, we assume that χ_0 is the unit of $\widehat{G}$. To investigate this case we will study properties of some family of functions $\mathcal{A}=\{f(g)\}$, $g\in G$, defined on G, which is uniformly bounded, equicontinuous and uniformly almost periodic. In this connection, we call a family of almost periodic on G functions uniformly almost periodic if for every $\varepsilon>0$ there exists a compact set $C\subset G$ such that every set zC, $z\in G$, contains an ε-translational element τ common for all $f\in\mathcal{A}$.

Lemma 4. *Let $\mathcal{A}=\{f(g)\}$ be a family of uniformly bounded equicontinuous and uniformly almost periodic functions on a locally compact group G. Then for every $\varepsilon>0$ there exists $\eta=\eta(\varepsilon)$ (not depending on a function of the family) such that $M\{|f(g)|^2\}<\eta$ implies* $\sup_{g\in G}|f(g)|^2<\varepsilon$ *for all $f\in\mathcal{A}$.*

Proof. This statement for the case $G = \mathbb{R}$ is proved in [1] (P. 48). To investigate the general case we need some modification. At first, we will prove that the family $\mathcal{A}$ is compact. Due to of equicontinuity of the family, for every $\varepsilon > 0$ there exists a neighborhood G_δ, $\delta = \delta(\varepsilon)$, of the unit such that

$$|f(g_1) - f(g_2)| < \varepsilon/5 \tag{10}$$

for all $f \in \mathcal{A}$, and $g_1, g_2 \in zG_\delta$, $z \in G$. Let C be a compact subset of G such that zC, $z \in G$, contains an $\varepsilon/5$-translation of τ_z for all $f \in \mathcal{A}$, i.e., $|f(g\tau_z) - f(g)| < \varepsilon/5$ for all $g \in G$ and $f \in \mathcal{A}$. Now we cover C by a finite number of sets $y_1G_\delta, y_2G_\delta, \dots, y_nG_\delta$ where $y_1, y_2, \dots, y_n \in C$. We choose a sequence in $\mathcal{A}$

$$f_1(g), f_2(g), \dots, \tag{11}$$

which converges at every point y_i, $i = 1, 2, \dots, n$. Next we fix a number $N = N(\varepsilon/5)$ such that for all $r, s > N$ the inequalities

$$|f_r(y_i) - f_s(y_i)| < \varepsilon/5, \;\; i = 1, 2, \dots, n, \tag{12}$$

hold. Consider an arbitrary element $g_0 \in G$. Now in the set $g_0^{-1}C$ we find an $\varepsilon/5$-translation of τ, common for all $f \in \mathcal{A}$. The element $y_0 = g_0\tau$ belongs to some set y_iG_δ. According to (10) and (12), taking into account convergence of the sequence (11), we obtain

$$\begin{aligned}|f_r(g_0) - f_s(g_0)| \le |f_r(g_0) - f_r(g_0\tau)| + |f_r(g_0\tau) - f_r(y_i)| \\ + |f_r(y_i) - f_s(y_i)| + |f_s(y_i) - f_s(g_0\tau)| + |f_s(g_0\tau) - f_s(g_0)| < \varepsilon.\end{aligned}$$

This implies compactness of $\mathcal{A}$.

Let $\varphi(g)$ be a non-negative almost periodic function on G such that $\varphi(g_0) \ge a > 0$ for some $g_0 \in G$. Let $\varphi(ga_1), \varphi(ga_2), \dots, \varphi(ga_n)$ be an $a/2$-net for the family $\{\varphi(gb)\}$ (g is fixed and b goes over G). This means that for every $b \in G$ there exists an element $a_i \in G$, $i \in \{1, 2, \dots, n\}$, such that $|\varphi(gb) - \varphi(ga_i)| < a/2$. For $g = g_0a_i^{-1}$ we have $\varphi(g_0a_i^{-1}b) > a/2$, therefore, for every $b \in G$

$$n^{-1}\sum_{i=1}^{n} \varphi(g_0a_i^{-1}b) > a/(2n).$$

From this inequality, taking into account known properties of mean value M ([6], P. 262) we obtain

$$M_b\{n^{-1}\sum_{i=1}^{n} \varphi(g_0a_i^{-1}b)\} = n^{-1}\sum_{i=1}^{n} M_b\{\varphi(g_0a_i^{-1}b)\} = n^{-1}\sum_{i=1}^{n} M_b\{\varphi(b)\} = M_b\{\varphi(b)\} > a/(2n). \tag{13}$$

Now assume that the contrary is valid. Then there exist $\varepsilon_0 > 0$, a decreasing sequence of numbers $\eta_1, \eta_2, \dots$, converging to zero, and an infinite sequence of functions $f_1(g), f_2(g), \dots$ from $\mathcal{A}$ such that

$$\sup_{g\in G} |f_m(g)|^2 > \varepsilon_0$$

and

$$M_g\{|f_m(g)|^2\} < \eta_m \tag{14}$$

for $m = 1,\ 2, \dots$ Consider the family $\mathcal{B}(G) = \{|f(g|^2\},\ f \in \mathcal{A}(G)$. For functions $\varphi_m(g) =: |f_m(g)|^2$ from (13) we obtain

$$M\{|f_m|^2\} > \varepsilon_0/(2n(\varepsilon_0))$$

which contradicts (14). □

Theorem 2. *Let the characters, appearing in the Fourier expansion* (5) *of* $f \in Q(G)$*, has a unique limiting point which is the unit character. If* $\alpha \in I$, $T_\alpha, K_\alpha \in U_{\widehat{G}}$, $T_\alpha, K_\alpha \to 0$ *as* $\alpha \to \infty$*, the functions* f_{T_α,K_α} *are defined by* (2) *and*

$$(\operatorname{mes} T_\alpha)^{-1} \operatorname{mes}(T_\alpha K_\alpha) \le C_1, \;\; \alpha \in I,$$

where C_1 does not depend on T_α and K_α, then

$$f_{T_\alpha,K_\alpha}(g) \to M\{f(g)\} = A_0, \ \ \alpha \to \infty \tag{15}$$

uniformly by $g \in G$.

Proof. From

$$(\operatorname{mes} T_\alpha)^{-1} \int_G |\widehat{(1)}_{T_\alpha}(h)\widehat{(1)}_{T_\alpha K_\alpha}(h)|dh \le (\operatorname{mes} T_\alpha)^{-1}\|\widehat{(1)}_{T_\alpha}\|_2\|\widehat{(1)}_{T_\alpha K_\alpha}\|_2$$
$$= (\operatorname{mes} T_\alpha)^{-1}\|(1)_{T_\alpha}\|_2\|(1)_{T_\alpha K_\alpha}\|_2 = (\operatorname{mes} T_\alpha)^{-1}\operatorname{mes}(T_\alpha K_\alpha) \le C_1,$$

and

$$|f_{T_\alpha,K_\alpha}(\tau g) - f_{T_\alpha,K_\alpha}(\tau g)| \le C_1 \sup_{g\in G}|f(\tau g) - f(g)|$$

it follows that the family f_{T_α,K_α} is uniformly bounded, equicontinuous and uniformly almost periodic. Applying the Parseval identity, Lemma 1 and the properties of the kernel of the operator f_{T_α,K_α}, mentioned above, we obtain

$$M\{|f_{T_\alpha,K_\alpha}(g) - A_0|^2\} = \sum_{\chi_k\in K_\alpha}{}' |A_k|^2 + \sum_{\chi_k\in T_\alpha K_\alpha\setminus K_\alpha}|A_k|^2\varphi_{T_\alpha,K_\alpha}(\chi_k);$$

here the prime in the first sum means that we omit the term $|A_0|^2$. It is clear that the expression in the right-hand side tends to zero as $\alpha \to \infty$. From this and Lemma 4 we obtain (15). □

Let all the characters χ_n, appearing in the Fourier expansion (5) of $f \in Q(G)$, belong to a set $U \in U_{\widehat{G}}$, and $f_{T,U}$, $T \in U_{\widehat{G}}$, be defined by (2). Similarly to Lemma 1 we prove that f and $f_{T,U}$ have the same Fourier series, therefore, by the known uniqueness theorem (see [6] or [1], P. 276), they coincide, i.e.,

$$f(g) = f_{T,U}(g) = (\operatorname{mes} T)^{-1}\int_G f(hg)\widehat{(1)}_T(g)\widehat{(1)}_{TU}(g)dh, \ \ g \in G.$$

Replacing in the right-hand side of the equality the Fourier transforms of the characteristic functions of the sets T and TU by their Laplace transforms [11], we obtain a defined on $G \times \operatorname{Hom}(\widehat{G},\mathbb{R})$ function

$$F(g,r) = (\operatorname{mes} T)^{-1}\int_G f(hg)L((1)_T)(r,h)L((1)_{TU})(r,h)dh \tag{16}$$

where, e.g., $L((1)_T)(r,h) = \int_G \chi(g)e^{-r(\chi)}d\chi$, and r is a real character of $\widehat{G}$. The integral in the right-hand side of (16) exists at every fixed point of $G \times \operatorname{Hom}(\widehat{G},\mathbb{R})$. Indeed,

$$|F(g,h| \le (\operatorname{mes} T)^{-1}\sup_{g\in G}|f(g)|\,\|\widehat{(e^{-r})}_T\|_2\|\widehat{(e^{-r})}_{TU}\|_2$$
$$= (\operatorname{mes} T)^{-1}\sup_{g\in G}|f(g)|\,\|(e^{-r})_T\|_2\|(e^{-r})_{TU}\|_2$$
$$\le \sup_{g\in G}|f(g)|\sup_{\chi\in U}e^{-r(\chi)}\sup_{\chi\in TU}e^{-r(\chi)}(\operatorname{mes}(TU)/\operatorname{mes} T)^{1/2}$$

where, e.g.,

$$(e^{-r})_T(\chi) = \begin{cases} e^{-r(\chi)}, & \chi \in T; \\ 0, & \chi \notin T. \end{cases}$$

The function $F(g,r)$, defined by (16), is remarkable that it is an extension of $f_{T,U}$, therefore, as well as f, from G to $G \times \operatorname{Hom}(\widehat{G},\mathbb{R})$. In the case $G = \mathbb{R}$ the function $F(g,r)$ is an analytic continuation of a uniform almost periodic on $\mathbb{R}$ function f to the whole complex plane $\mathbb{C}$. The problem of extension of functions, defined on a locally compact group G, belonging to some class, from G to $G \times \mathrm{Hom}(\widehat{G}, \mathbb{R})$, was considered in [12].

Now we will investigate the problem of summability of Fourier series in the case when there is no any additional restrictions on the set of characters $\chi_n(g)$, $n \in N$, appearing in (5).

A finite or countable set of characters $B(g) = \{\beta_1(g), \beta_2(g), \dots\}$ we call a basis of a sequence of characters $\{\chi_n(g)\}$, if it satisfies the following properties:

1) the characters $\beta_1, \beta_2, \dots$ are independent in the sense that there are no relations of the type

$$\beta_1^{a_1}(g)\beta_2^{a_2}(g)\dots\beta_m^{a_m}(g) \equiv 1, \;\; g \in G, \;\; m = 1, 2, \dots,$$

with integer numbers $a_1, a_2, \dots, a_m$ not equal zero simultaneously (if, e.g., $a_k < 0$, then $\beta_k^{a_k}(g) = \beta_k^{-a_k}(g^{-1})$);

2) for every character $\chi \in \{\chi_n\}$ there exists a set of integers $r_1, r_2, \dots, r_m, r$ such that

$$\chi^r(g) = \beta_1^{r_1}(g)\beta_2^{r_2}(g)\dots\beta_m^{r_m}(g). \tag{17}$$

Here without loss of generality we assume that the numbers $r, r_1, r_2, \dots, r_m$ have no common divisors, i.e., $d := (r, r_1, \dots, r_m) \neq 1$.

For a given character χ representation (17) is unique. Indeed, assume that for χ there are two representations $\chi^r = \beta_1^{r_1}, \dots, \beta_m^{r_m}$ and $\chi^s = \beta_1^{s_1}, \dots, \beta_m^{s_m}$ where $(r, r_1, \dots, r_m) \neq 1$ and $(s, s_1, \dots, s_m) \neq 1$. Then, by property 1), we have $r_k s - r s_k = 0$, $k = 1, \dots, m$, i.e. $r_k/s_k = r/s = a/b$ where the fraction a/b is irreducible. We obtain that all the numbers s and s_k, $k = 1, \dots, m$, are divisible by b, but this is not valid by our assumption. Thus, representation (17) is unique. It is also easy to prove that if for some two characters χ_1 and χ_2 we have $\chi_1^r = \beta_1^{r_1}, \dots, \beta_m^{r_m}$, $\chi_2^s = \beta_1^{s_1}, \dots, \beta_m^{s_m}$, and $r_k/r = s_k/s$, $k = 1, \dots, m$, then $\chi_1 = \chi_2$.

By a simple modification of the known method of constructing basic systems of numbers for sequences of real numbers (see, e.g., [1], P. 67) we can create a basis B for every sequence of characters $\{\chi_n\}$.

Below we introduce some analog of the known Bochner–Fejer means of Fourier series; for the case $G = R$ it is studied, e.g., in [1] (P. 70), [13, 14].

We consider a fixed basis $B(g) = \{\beta_1(g), \beta_2(g), \dots\}$ for a sequence of characters $\{\chi_n\}$, appearing in the Fourier expansion of a function $f \in Q(G)$. For convenience, we assume that property 2) is valid for every character $\chi \in \widehat{G}$ (where r, $r_1, \dots, r_m$ are arbitrary integers). In the opposite case we equate the Fourier coefficient at χ to zero.

Let q be an arbitrary sufficiently large natural number and $qq! = P$. We denote by $E_q := E_{q,\beta_1,\dots,\beta_q}$ the set of all characters χ_n from $U_{\widehat{G}}$ which can be represented as

$$\chi_n^{q!}(g) = \beta_1^{\nu_1}(g)\beta_2^{\nu_2}(g)\dots\beta_q^{\nu_q}(g) \tag{18}$$

where $\beta_1(g), \beta_2(g), \dots, \beta_q(g)$ are functions from $B(g)$, and the integers $\nu_1, \nu_2, \dots, \nu_q$ satisfy $|\nu_k| \le P$, $k = 1, 2, \dots, q$. It is clear that $E_1 \subset E_2 \subset \dots \subset E_q$. If the numbers $\nu_k/q!$ are written in the irreducible form $\nu_k/q! = s_k/t_k$, $k = 1, \dots, q$, where t_k are positive,

$$q_0 = \max\{m, t_1, \dots, t_m, |s_1|/|t_1|, \dots, |s_m|/t_m\},$$

then the characters χ_n, defined by (18), are contained in E_q for all $q > q_0$. It is clear that if $\chi \in E_q$, then the inverse to χ character $\chi^{-1}(g) = \chi(g^{-1}) = \overline{\chi(g)}$ belongs to the same set.

Now we compose the following kernel of Fejer–Bochner type:

$$K^{(q)}(g) := K^{(q)}_{\beta_1,\dots,\beta_q}(g) = \sum_{\nu_1=-P}^{P}\sum_{\nu_2=-P}^{P}\dots\sum_{\nu_q=-P}^{P}(1 - |\nu_1|/P)(1 - |\nu_2|/P)\dots(1 - |\nu_q|/P)\chi_n(g) \tag{19}$$

where $\chi_n \in E_q$ is the character corresponding to the set $\nu_1, \dots, \nu_q, q$ by (18). The sum (19), together with χ_n, also contains its inverse character $\chi_n^{-1}(g)$. We rewrite (19) in the form $K^{(q)}(g) = \sum k_n^{(q)}\chi_n(g)$ where $k_n^{(q)} = (1 - |\nu_1|/P)(1 - |\nu_2|/P)\dots(1 - |\nu_m|/P)1\dots1$, $|\nu_k/P| = |r_k|/qr$, $k = 1, \dots, m$, and the number of units in the product $1\dots1$ equals $q-m$. We note that $k_n^{(q)}=0$, if χ_n does not belong to E_q. From this it is clear that $0\le k_n^{(q)}\le 1$ for all n, q, and $k_n^{(q)}\to 1$ for fixed n and $q\to\infty$. Every character β_j, $1\le j\le q$, from (18) is represented in the form $\beta_j(g)=e^{i\xi_j(g)}$ where ξ_j is a continuous function defined on G such that $\xi_j(g_1g_2)=\xi_j(g_1)+\xi_j(g_2)$, $g_1,g_2\in G$. Then

$$K^{(q)}(g)=\prod_{j=1}^{q}\Big\{\sum_{\nu_j=-P}^{P}(1-|\nu_j|/P)e^{\xi_j(g)\nu_j/q!}\Big\}=P^{-1}\prod_{j=1}^{P}(\sin^2(\xi_j(g)/2))^{-1}\sin^2(P\xi_j(g)/2)\ge 0.$$

Now we compose an analog of the known Fejer–Bochner means for locally compact Abelian groups G,

$$S^{(q)}(f,g)=M_t\{f(gt)K^{(q)}(t)\},\quad g,t\in G, \tag{20}$$

where the kernels $K^{(q)}$ are defined by (19) and M_t is the averaging (with respect to $t\in G$) operator in G. Positiveness of $K^{(q)}$ and known properties of M imply the uniform estimate

$$|S^{(q)}(f,g)|\le\sup_{g\in G}|f(g)|. \tag{21}$$

For $G=\mathbb{R}$ we obtain from (20) the classical Fejer–Bochner polynomials.

For representation (20) of means in the form

$$S^{(q)}(f,g)=\sum k_n^{(q)}A_n(f)\chi_n(g),\quad g\in G, \tag{22}$$

from the relation $f(gt)\sim\sum A_n(f)\chi_n(gt)=\sum A_n(f)\chi_n(g)\chi_n(t)$ and the equality

$$A_n(f)=M\{f(t)\overline{\chi_n(t)}\}=M_t\{f(gt)\overline{\chi_n(t)}\}$$

([1], Chap. 6, § 2), we obtain

$$S^{(q)}(f,g)=\sum k_n^{(q)}M_t\{f(gt)\overline{\chi_n(t)}\}=M_t\{f(gt)\sum k_n^{(q)}\overline{\chi_n(t)}\}=M_t\{f(gt)K^{(q)}\}.$$

Theorem 3. *Let f be a continuous almost periodic on a locally compact Abelian group G function with Fourier series* (5)*. Then, uniformly by $g\in G$, the equality $\lim_{q\to\infty}S^{(q)}(f,g)=f(g)$ holds.*

Proof. Consider an arbitrary trigonometric polynomial

$$Q(g):=Q_E(g)=\sum_{n=1}^{N}b_n\chi_n(g)=\sum_{n=1}^{N}A_n(Q_E)\chi_n(g)$$

of fixed degree $E\in\widehat{G}$ where the characters $\chi_n\in E$ are taken from the Fourier expansion (5) of f. According to (22), we obtain

$$S^{(q)}(Q,g)=\sum_{\chi_n\in E}k_n^{(q)}A_n(Q)\chi_n(g).$$

Assuming $\lim\limits_{q\to\infty}k_n^{(q)}=1$, we make sure that Theorem 3 is valid for $f=Q$.

Let now f be an arbitrary function satisfying the assumptions of Theorem 3. By the approximation theorem, for every $\varepsilon>0$ there exists a polynomial Q_ε which is a linear combination of the characters, appearing in the Fourier expansion of f, such that

$$|f(g)-Q_\varepsilon(f,g)|\le\varepsilon. \tag{23}$$

From (21), (22), and the properties of kernels (19) mentioned above it follows that

$$\sup_{g\in G}|S^{(q)}(f,g)-S^{(q)}(Q_\varepsilon,g)|=\sup_{g\in G}|S^{(q)}(f-Q_\varepsilon)(g)|\le\sup_{g\in G}|f(g)-Q_\varepsilon(g)|\le\varepsilon. \tag{24}$$

As we proved earlier, Theorem 3 is valid for polynomial Q_ε, therefore, for a fixed ε there exists sufficiently large number $q_0(\varepsilon)$ such that

$$\sup_{g\in G}|Q_\varepsilon(f,g)-S^{(q)}(Q_\varepsilon,g)|<\varepsilon \tag{25}$$

for $q\geq q_0$. From

$$|f(g)-S^{(q)}(f,g)|\leq|f(g)-Q_\varepsilon(f,g)|+|Q_\varepsilon(f,g)-S^{(q)}(Q_\varepsilon,g)|+|S^{(q)}(Q_\varepsilon,g)-S^{(q)}(f,g)|$$

and (23)–(25) Theorem 3 follows. □

We can obtain a similar theorem for means of the form (20), which are constructed on the base of elliptic trigonometric Fejer polynomials considered in [15]. For convenience, we give their definition. Let $(u,Au)=\sum\limits_{k,l=1}^{m}a_{k,l}u_ku_l$ be a positive definite quadratic form, $A=(a_{k,l})_{k,l=1}^m$ be corresponding symmetric matrix, and $d(x)=(u,Au)^{1/2}$. We denote by $S_r(u_0)$ the set $\{u:d(u-u_0)\leq r\}$. An elliptic trigonometric Fejer polynomial (of degree q) of variable $x\in\mathbb{R}^m$ has the form $\widetilde{k}^{(q)}(x)=\sum\limits_{d(\nu)\leq q}\lambda_\nu e^{i(\nu,x)}$, $\lambda_\nu=(\operatorname{mes}S_{q/2}(0))^{-1}\operatorname{mes}(S_{q/2}(0)\cap S_{q/2}(\nu))$ where $x\in\mathbb{R}^m$. In [15] it is proved that $\widetilde{k}^{(q)}(x)\geq 0$, $x\in\mathbb{R}^m$. With the help of them, we construct the kernels

$$\widetilde{K}^{(q)}(g):=\sum_{d(\nu)\leq P}\lambda_\nu^{(q)}\chi_n(g),\quad P=qq!.$$

Positiveness of elliptic trigonometric Fejer polynomial implies the inequality $\widetilde{K}^{(q)}(x)\geq 0$. Let

$$\widetilde{S}^{(q)}(f,g)=M_t\{f(gt)\widetilde{K}^{(q)}(t)\},\quad g,t\in G.$$

Repeating the proof of Theorem 3, we make sure that there is valid

Theorem 4. *If $f\in Q(G)$, then $\lim_{q\to\infty}\widetilde{S}^{(q)}(f,g)=f(g)$ uniformly on G.*

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Universal Computable Enumerations of Finite Classes of Families of Total Functions

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Abstract—In the paper we introduce the notion of a computable enumeration of a class of families. We prove a criteria for the existence of universal computable enumerations of finite classes of computable families of total functions. In particular, we show that there is a finite computable class of families of total functions without universal computable enumerations.

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Introduction. Nowadays, there is a large number of papers devoted to generalized-computable enumerations. These include a fundamental in this regard paper by S. Goncharov and A. Sorbitol [1], papers devoted to investigations of arithmetical enumerations (for example, [2–4]), and a recent paper by S. Badaev and S. Goncharov [5], where generalized-computable enumerations are considered from the standpoint of uniform enumerability of families relative to an arbitrary oracle. The goal of this paper is to generalize the concept of computable enumeration up to computability of countable classes of families. This is connected with the notion of computable enumerations of partial functionals of finite types introduced in [6]. We consider the questions mainly concerning the universal computable enumerations. It is known [6] that in a classic case, any finite family of computably-enumerable (c.-e.) sets has a universal computable enumeration. The main purpose of this work is a search of a criterion of the existence of universal enumerations of finite computable classes of families of total functions, as well as a search of a class that does not have any universal computable enumerations.

Necessary information on the numeration theory can be found in the monograph by J. L. Yershov [6]. An *enumeration* of a nonempty set X is an arbitrary surjection $\nu : \mathbb{N} \to X$. Let ν be an enumeration of X, μ be an enumeration of $Y \subseteq X$. We say that μ *is reduced* to ν ($\mu \leqslant \nu$) if there exists a computable function f such that $\mu = \nu \circ f$. We denote $\nu \equiv \mu$ if $\mu \leqslant \nu$ and $\nu \leqslant \mu$. A *concatenation* of enumerations ν_0 and ν_1 is an enumeration $\nu_0 \oplus \nu_1(2x + i) = \nu_i(x)$, where $i = 0, 1$. Let $\mathcal{F}$ be a countable family of subsets of $\mathbb{N}$. An enumeration ν of the family $\mathcal{F}$ is called *computable* if a set $G_\nu = \{\langle x, y\rangle : y \in \nu x\}$ is c.-e. If any computable enumeration $\mathcal{F}$ is reduced to ν, then ν is called *universal*. A set of computable enumerations of the family $\mathcal{F}$, factorized under the relation $\equiv$, forms an upper semilattice with respect to the operation $[\nu] \vee [\mu] = [\nu \oplus \nu]$. This semilattice is called the *Rogers semilattice* of the family $\mathcal{F}$ and is denoted by $\mathcal{L}^0(\mathcal{F})$.

Concepts of n -families and their enumerations were introduced in [7–9]. So, 0-*family* is an arbitrary subset of $\mathbb{N}$, $(n + 1)$-*family* is a set whose elements are m-families, $m \leqslant n$. In particular, classes of families are 2-families. Using the notion of enumeration of n-families, a concept of their computable enumeration is naturally introduced. Because in this paper we examine only classes of families, we confine ourselves to the case of 2-families whose elements are 1-families. Let $\mathcal{F}$ be a class of families and ν be its enumeration.

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Definition. An enumeration ν is called *computable* if there exists a partially computable function f such that for each x

$$\nu x = \{W_{f(x,y)} : f(x,y)\downarrow \ \& \ y \in \mathbb{N}\}.$$

If elements of the class $\mathcal{F}$ are families of total functions, then in Definition 22 $W_{f(x,y)}$ can be replaced by $\varphi_{f(x,y)}$. As in the case of 1-families, the set of all computable enumerations of the given class $\mathcal{F}$, factorized under the relation $\equiv$, forms an upper semilattice with respect to the operation $\oplus$. We will denote this upper semilattice by $\mathcal{L}^0_1(\mathcal{F})$ and as well will call it *Rogers semilattice* of the class of families $\mathcal{F}$. It is not difficult to see that for each computable family $\mathcal{A}$ one can find a class of families of total functions $\mathcal{F}$ such that the Rogers semilattice of this class is isomorphic to $\mathcal{L}^0(\mathcal{A})$. Indeed, for each $X \in A$ it suffices to consider a family

$$\mathcal{F}_X = \{f : f(0) \in X \ \& \ \forall x > 0\ [f(x) = 0]\}$$

and to set $\mathcal{F} = \{\mathcal{F}_X : X \in \mathcal{A}\}$.

Main results. Let $\mathcal{A}$ be a Σ^0_2-computable family and $\mathcal{L}^1(\mathcal{A})$ be the Rogers semilattice of its Σ^0_2-computable enumerations. As in the case of computable families, $\mathcal{L}^1(\mathcal{A})$ is isomorphic to the Rogers semilattice of the class of families of total functions.

Proposition 1. *There exists a computable class $\mathcal{F}$ of families of total functions such that $\mathcal{L}^0_1(\mathcal{F}) \cong \mathcal{L}^1(\mathcal{A})$.*

Sketch of the proof. For each $X \in \mathcal{A}$, we consider a family of functions

$$\mathcal{F}_X = \{f : \exists n\, \forall m \geqslant n\ [f(m) = 0]\} \cup \{f : f(0) \in X \ \& \ \exists n\, \forall m \geqslant n\ [f(m) = 1]\}.$$

Let $\mathcal{F} = \{\mathcal{F}_X : X \in \mathcal{A}\}$. We show that the correspondence $\nu \mapsto \widehat{\nu}$, where $\widehat{\nu}x = \mathcal{F}_{\nu x}$ induces an isomorphism from $\mathcal{L}^1(\mathcal{A})$ onto $\mathcal{L}^0_1(\mathcal{F})$. It is easy to see that for arbitrary Σ^0_2-computable enumerations μ, ν of the family $\mathcal{A}$

$$\mu \leqslant \nu \Leftrightarrow \widehat{\mu} \leqslant \widehat{\nu}.$$

Let us show that for each Σ^0_2-computable enumeration ν of the family $\mathcal{A}$, $\widehat{\nu}$ is a computable enumeration of the class $\mathcal{F}$. We fix a computable function g such that for all x, y

$$\langle x, y\rangle \in G_\nu \Leftrightarrow \liminf_s g(s,x,y) = 1.$$

Let $\{f_k\}_{k\in\mathbb{N}}$ be a uniformly computable sequence of all functions with finite support, $\{h^z_k\}_{z,k\in\mathbb{N}}$ be a uniformly computable sequence of functions satisfying the following properties

$$h^z_k(0) = z,\ \exists n\, \forall m \geqslant n\ [h^z_k(m) = 1].$$

Let us define a computable function f, supposing

$$\varphi_{f(x,y)}(i) = \begin{cases} f_k(i), & \text{if } y = 2k;\\ h^z_k(i), & \text{if } y = 2\langle k,z,s\rangle + 1 \text{ и } \forall v\ [s \leqslant v \leqslant i \Rightarrow g(v,x,z) = 1];\\ 0 & \text{otherwise.}\end{cases}$$

By definition of f, we have

$$\widehat{\nu}x = \{\varphi_{f(x,y)} : y \in \mathbb{N}\}$$

for all x. Now let μ be an arbitrary computable enumeration of the class $\mathcal{F}$. We choose a computable function h such that

$$\mu x = \{\varphi_{h(x,y)} : y \in \mathbb{N}\}$$

for all x. Then for a Σ^0_2-computable enumeration ν of the family $\mathcal{A}$, which is determined by the following equation

$$\nu x = \{y : \exists z\, \exists n\, \forall m \geqslant n\ [\varphi_{h(x,z)}(0) = y \ \& \ \varphi_{h(x,z)}(m) = 1]\},$$

we have $\mu = \widehat{\nu}$.

In Proposition 1, all families of the class $\mathcal{F}$, considered as subsets of the Baire space, have equal closures. For finite classes $\mathcal{F}$ the converse statement also holds: If closures of all families of a given class $\mathcal{F}$ are equal, then $\mathcal{L}^0_1(\mathcal{F})$ is isomorphic to the Rogers semilattice of some finite family of Σ^0_2-sets. Moreover, we have the following.

Proposition 2. *Let $\mathcal{F}$ be a finite class of computable families of total functions, whose closures coincide, $\mathcal{A}$ be a finite family of Σ^0_2-sets. If $(\mathcal{F}, \subseteq) \cong (\mathcal{A}, \subseteq)$, then $\mathcal{L}^0_1(\mathcal{F}) \cong \mathcal{L}^1(\mathcal{A})$.*

Sketch of the proof. We assume that $\emptyset \notin \mathcal{F}$ since otherwise the proposition is trivial. Let

$$\mathcal{A} = \{A_0, A_1, \dots, A_n\},\ \mathcal{F} = \{\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_n\}$$

and $A_i \subseteq A_j$ if and only if $\mathcal{F}_i \subseteq \mathcal{F}_j$. We fix a sequence of finite sets $F_0, F_1, \dots, F_n$ such that for all $i, j \leqslant n$

1) $F_i \subseteq A_i$,

2) $F_i \subseteq A_j \Rightarrow A_i \subseteq A_j$,

3) $A_i \subseteq A_j \Rightarrow F_i \subseteq F_j$.

Since each of the families $\mathcal{F}_i$, $i \leqslant n$, is computable, one can fix a uniformly computable with respect to i, j sequence of functions f_{ij} such that

$$\mathcal{F}_i = \{f_{ij} : j \in \mathbb{N}\}.$$

We show that the correspondence $\nu \mapsto \widehat{\nu}$, where $\nu x = A_i \Leftrightarrow \widehat{\nu} x = \mathcal{F}_i$, induces an isomorphism from $\mathcal{L}^1(\mathcal{A})$ onto $\mathcal{L}^0_1(\mathcal{F})$. By definition, $\mu \leqslant \nu \Leftrightarrow \widehat{\mu} \leqslant \widehat{\nu}$. First, we prove that each of the enumerations $\widehat{\nu}$ is computable. Since a Σ^0_2-index of the set νx can be found uniformly in respect to x, there exists a computable function $h : \mathbb{N}^2 \to \{0, \dots, n\}$ such that $\nu x = A_i$ if and only if

a) $\exists^\infty s\, [h(x, s) = i]$,

b) $\exists s \forall t \geqslant s\, [h(x, s) = i \,\&\, F_i \subseteq F_{h(x,t)}]$.

Define a computable function f such that

$$\widehat{\nu} x = \{\varphi_{f(x,y)} : y \in \mathbb{N}\}.$$

Let $d(x, t, s) = 1 + t + \max\{v < s : 0 < v \,\&\, F_{h(x,t+v)} \nsubseteq F_{h(x,t+v+1)}\}$ (we suppose that $\max \emptyset = 0$). We set

$$\varphi_{f(x,\langle y,t\rangle)}(s) = \begin{cases} f_{h(x,t)y}(s), & \text{if } \forall v \leqslant s\, [F_{h(x,t)} \subseteq F_{h(x,t+v)}]; \\ f_{h(x,d(x,t,s))z}(s) & \text{otherwise, where} \\ & z = \min\{u : \forall v < s\, [\varphi_{f(x,\langle y,t\rangle)}(v) = f_{h(x,d(x,t,s))u}(v)]\}. \end{cases}$$

Since closures of all families $\mathcal{F}$ are equal, we have that such z always exists.

Now let μ be an arbitrary computable enumeration of the class $\mathcal{F}$ and f be a computable function, for which $\mu x = \{\varphi_{f(x,y)} : y \in \mathbb{N}\}$. Let us fix a sequence of finite families of functions $\mathcal{H}_i$, $i \leqslant n$, satisfying properties 1)–3) with replacing F_i by $\mathcal{H}_i$ and A_i by $\mathcal{F}_i$. Consider an enumeration ν of the family $\mathcal{A}$, defined by the following property

$$\nu x = A_i \Leftrightarrow \mu x = \mathcal{F}_i.$$

We show that ν is Σ^0_2-computable. Indeed,

$$y \in \nu x \Leftrightarrow \exists i\, [y \in A_i \,\&\, A_i \subseteq \nu x],$$

$$A_i \subseteq \nu x \Leftrightarrow \mathcal{F}_i \subseteq \mu x \Leftrightarrow \mathcal{H}_i \subseteq \mu x \Leftrightarrow \exists k,\ \exists y_0, \dots, y_k\, [\mathcal{H}_i = \{\varphi_{f(x,y_j)} : j \leqslant k\}].$$

So, ν is Σ_2^0-computable and $\mu = \widehat{\nu}$.

By using the criterion of the existence of universal arithmetical enumerations of finite families from [3] and Proposition 2 we obtain the appropriate criterion for finite classes of computable families of total functions $\mathcal{F}$ having equal closures. Namely, $\mathcal{F}$ has a universal computable enumeration if and only if it contains the least family with respect to inclusion. Now let $\mathcal{F}$ be an arbitrary finite class of computable families of total functions.

Theorem. *$\mathcal{F}$ has a universal computable enumeration if and only if each of its subclass, consisting of all the families with equal closures, has the least family with respect to inclusion.*

Let us present the sketch of the proof of the theorem. Let

$$\mathcal{F} = \{\mathcal{F}_0^0, \dots, \mathcal{F}_{N_0}^0; \mathcal{F}_0^1, \dots, \mathcal{F}_{N_1}^1; \dots; \mathcal{F}_0^k, \dots, \mathcal{F}_{N_k}^k\},$$

$\overline{\mathcal{F}}_0^i \neq \overline{\mathcal{F}}_0^j$ when $i < j \leqslant k$, $\overline{\mathcal{F}}_j^i = \overline{\mathcal{F}}_l^i$ when $i \leqslant k$ and $j, l \leqslant N_i$. By using Proposition 2 and the mentioned criterion from [3] we show that $\mathcal{F}$ has a universal enumeration if and only if each of the classes $\{\mathcal{F}_0^i, \dots, \mathcal{F}_{N_i}^i\}$, $i \leqslant k$, has a universal enumeration. Let each of the classes $\mathcal{C}_i = \{\mathcal{F}_0^i, \dots, \mathcal{F}_{N_i}^i\}$ have a universal enumeration μ_i. We assume that $\mathcal{F}_0^i$ is the least family of $\mathcal{C}_i$ under inclusion. We fix a sequence $F_0, \dots, F_k$ of finite subsets of $\mathbb{N}^{<\mathbb{N}}$, satisfying the following properties for all $i, j \leqslant k$

i) $\forall \sigma \in F_i \left[[\sigma] \cap \mathcal{F}_0^i \neq \varnothing\right]$,

ii) $\forall \sigma \in F_i \left[[\sigma] \cap \mathcal{F}_0^j \neq \varnothing\right] \Rightarrow \overline{\mathcal{F}}_0^i \subseteq \overline{\mathcal{F}}_0^j$,

iii) $\overline{\mathcal{F}}_0^i \subseteq \overline{\mathcal{F}}_0^j \Rightarrow F_i \subseteq F_j$,

where $[\sigma]$ denotes a base neighborhood in $\mathbb{N}^{\mathbb{N}}$ generated by the string σ. Let $\{\alpha_e\}_{e \in \mathbb{N}}$ be a uniform numbering of computable enumerations of all classes of families. For each $i \leqslant k$, we consider a c.-e. set

$$M_i = \{x : \forall \sigma \in F_i \left[[\sigma] \cap \alpha_e x \neq \varnothing\right] \text{ or } \alpha_e x \text{ is not a family of total functions}\}.$$

We fix a computable enumeration $\{x_j^i\}_{j \in \mathbb{N}}$ of the set M_i and define an enumeration α_e^i of the class $\mathcal{C}_i$ such that $\alpha_e^i n = \mathcal{F}_m^i$, $m \leqslant N_i$ if $\alpha_e x_n^i = \mathcal{F}_m^i$. Now we can define, uniformly with respect to a sequence of numbers $e, z_0, \dots, z_k$, an enumeration ν of the class $\mathcal{F}$ such that $\alpha_e \leqslant \nu$ if $\alpha_e^i = \mu_i \varphi_{z_i}$ for all $i \leqslant k$. The required universal enumeration is a concatenation of all such enumerations ν.

Conversely, let $\mathcal{F}$ have a universal computable enumeration μ. For each $i \leqslant k$, we consider a set

$$K_i = \{x : \forall \sigma \in F_i \left[[\sigma] \cap \mu x \neq \varnothing\right]\}.$$

We fix a computable enumeration $\{x_j^i\}_{j \in \mathbb{N}}$ of the set K_i and assume $\nu^i n$ to be equal μx_n^i if $\mu x_n^i \in \mathcal{C}_i$, and $\mathcal{F}_0^i$ otherwise. Since μ is a universal enumeration of $\mathcal{F}$ we have that ν^i is a universal for $\mathcal{C}_i$.

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