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ISOPERIMETRIC PROBLEMS FOR THE ROTATION FUNCTIONALS OF FIRST AND SECOND ORDER IN (PSEUDO)RIEMANNIAN MANIFOLDS

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In this paper we proceed with the investigation started in [1], [2]. We prove that the rotation Lagrangians of first and second order in (pseudo)Riemannian manifolds are singular, and consider various isoperimetric problems for these functionals.

1. Singularity of rotation functionals

The theory of variational problems with higher order derivatives for nondegenerate functionals goes back to M.V. Ostrogradskii's papers (see [3]). All efforts to develop an analog of this theory for variational problems on degenerate Lagrangians (see [4], [5]), which are important for applications in theoretical physics, run into obstacles connected mainly with the fact that the Euler-Lagrange equations for the extremals have no normal form. Therefore one cannot find an appropriate analog of the Hamilton equations with respect to the canonical variables. In this section we prove that the rotation Lagrangians of first and second order in (pseudo)Riemannian spaces of arbitrary dimension are degenerate.

Let (M^n, g) be a (pseudo)Riemannian manifold, and $\gamma :]t_0, t_1[\to M^n$ be a smooth curve. Using the covariant derivatives with respect to the Levi-Civita connection (the metric connection without torsion [6], p. 153), we construct vectors of higher curvatures $\underset{1}{\xi} = \nabla_t \xi, \dots, \underset{p}{\xi} = \nabla_t \underset{p-1}{\xi}$ along γ. Let us consider the length functional $\ell[\gamma] = \int_{t_0}^{t_1} \sqrt{e_0 \langle \dot\gamma, \dot\gamma\rangle}\, dt$, where $e_0 = \pm 1$ is the sign of the scalar square of tangent vector with respect to g. Let us also consider the rotation functionals $\underset{\alpha}{\theta}[\gamma] = \int_{\ell_0}^{\ell_1} k_\alpha(\ell) d\ell$ of order $\alpha = 1, \dots, n-1$. Here $\underset{\alpha}{k}(\ell)$ is the Frenet curvature (of order α) of γ ([7], p. 480). For an arbitrary parametrization of the curve, the Lagrangians of rotation functionals of first and second order are:

$$L_1(x, \xi, \underset{1}{\xi}) = e_0 \langle \xi, \xi\rangle^{-1} \sqrt{\varepsilon_1 G_1}, \quad L_2(x, \xi, \underset{1}{\xi}, \underset{2}{\xi}) = \varepsilon_1 G_1^{-1} \sqrt{e_0 \langle \xi, \xi\rangle}\, \sqrt{\varepsilon_2 G_2},$$

where $G_1 = \mathrm{Gr}(\xi, \underset{1}{\xi})$, $G_2 = \mathrm{Gr}(\xi, \underset{1}{\xi}, \underset{2}{\xi})$ are the Gram determinants of appropriate systems of vectors, ε_1, ε_2 are the signs of these determinants. Applying the operator of covariant differentiation with respect to the Levi-Civita connection, we have

$$\xi^i = \dot x^i, \quad \underset{1}{\xi}{}^i = \ddot x^i + \Gamma^i_{jk}(x) \dot x^j \dot x^k, \quad \underset{2}{\xi}{}^i = \dddot x^i + \partial_l \Gamma^i_{jk}(x) \dot x^j \dot x^k \dot x^l + 2\Gamma^i_{jk}(x) \dot x^j \ddot x^k + \Gamma^i_{jk}(x) \dot x^j \underset{1}{\xi}{}^k, \dots,$$