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SUBMANIFOLDS WITH SEMI-PARALLEL HIGHER ORDER FUNDAMENTAL FORMS AS ENVELOPES

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A submanifold M in a space of constant curvature $M_n(c)$ is said to be semi-symmetric if its second fundamental form α_2 satisfies the condition $\overline{\nabla}_X\overline{\nabla}_Y\alpha_2 = \overline{\nabla}_Y\overline{\nabla}_X\alpha_2$, where $\overline{\nabla}$ denotes the van der Waerden-Bortolotti connection [1]. In terms of the curvature operator $\overline{R}(X,Y) = [\overline{\nabla}_X, \overline{\nabla}_Y]$ of the connection $\overline{\nabla}$, the above condition, called the condition of semi-parallelity of α_2, can be represented in the following form $\overline{R}(X,Y)\cdot\alpha_2 = 0$.

One of the most important geometric characteristics of the semi-symmetric submanifolds is the following

Theorem 1 ([2]). *A submanifold M in $M_n(c)$ is semi-symmetric if and only if it is the second order envelope of a family of symmetric* (*in the sense of D. Ferus* [3]) *submanifolds.*

Analytically, the symmetric submanifolds are characterized by the condition of parallelity of the second fundamental form α_2, i.e., $\overline{\nabla}\alpha_2 = 0$ (see [3]).

A detailed survey on the results concerning semi-symmetric submanifold and comprehensive bibliography can be found in [4] (see also [5]–[8]).

Theorem 1 cited above can be generalized and developed in the directions of strengthening or weakening the law of enveloping. One of the possibilities to strengthen the law of enveloping was considered by the author in [9], where the following theorem was proved.

Theorem 2 ([9]). *Let an m-dimensional C^∞ submanifold M in an Euclidean space E_n have symmetric fundamental forms $\alpha_3, \alpha_4, \ldots, \alpha_{s-1}$ $(s \geq 4)$. Then the fundamental form α_s is symmetric if and only if M is the envelope of order $(s-2)$ of a family of m-dimensional submanifolds each possessing the following properties: its fundamental forms of all orders up to $(s-2)$ inclusively are symmetric and the fundamental form $\widetilde{\alpha}_{s-2}$ is, in addition, parallel, i.e., $\overline{\nabla}\widetilde{\alpha}_{s-2} = 0$.*

Theorem 1, as well as Theorem 2, are some partial cases of a general principle which can be stated as follows:

Theorem 3. *An m-dimensional C^∞ submanifold M in an Euclidean space E_n has semi-parallel fundamental forms α_{s-1} and α_s (for $s = 2$, only semi-parallel fundamental form α_2) if and only if it is the envelope of order s of a family of m-dimensional submanifolds each of which has parallel fundamental form of order s.*

The semi-parallelity of α_r means that $\overline{R}(X,Y)\cdot\alpha_r = 0$.

Theorem 3 can be proved by using the method of moving frame due to Cartan and the Frobenius–Cartan theory of completely integrable differential systems. The main ideas of the proof are same as those in [2] and [9].

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