Russian Mathematics (Iz. VUZ) *Izvestiya VUZ. Matematika*
Vol. 45, No. 5, pp. 19–27, 2001 *UDC 517.54*

AVERAGING TRANSFORMATION OF SETS AND FUNCTIONS ON THE RIEMANN SURFACES

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Averaging transformations of domains introduced by Marcus in [1] obtained significant application in the geometric theory of function of complex variable (see, e. g., [2]–[4]). Let us note the works [2], [3], where, in particular, generalizations of this transformation and numerous applications to various classes of analytic functions were given. In this article the Marcus transformation (see [1]) is developed in a different direction. We average sets lying on Riemann surfaces over the complex plane with regard for the multiplicity of sheets of the covering. Analogous transformations for circle symmetrization were realized in [5]–[8] and in other works. Ideologically, in this article we combine averaging in [1] and piecewise dividing symmetrization in [4]. However, transformation constructed here cannot be reduced to those mentioned above. In the capacity of applications we give new theorems on covering of segments, on mutually p-sheeted domains, and inequalities for rational functions.

1. Linearly averaging transformation

Let $\mathcal{R}$ be a Riemann surface lying over the complex plane $z = x + iy$, $\mathcal{R}(\Pi)$ a set of points of the surface $\mathcal{R}$, whose projections belong to the strip $\Pi = \{z : 0 \le x \le 1\}$. Speaking on open and closed sets on $\mathcal{R}(\Pi)$, we will mean the relative topology which is the trace on $\mathcal{R}(\Pi)$ of the topology given in $\mathcal{R}$. A collection of open sets $\{\mathcal{D}\}$ is called an admissible family of sets if the following conditions are fulfilled: Each set $\mathcal{D}$ belongs to the set $\mathcal{R}(\Pi)$ of a certain Riemann surface $\mathcal{R}$; the sets $\{\mathcal{D}\}$ are mutually p-sheeted, i. e., each point $z \in \Pi$ is covered by at most p distinct points belonging to the sets from the collection $\{\mathcal{D}\}$ (with regard for the multiplicity); a number $c > 0$ exists such that the sets $\{\mathcal{D}\}$ cover p-multiply the set $\Pi_c = \{z : 0 \le x \le 1,\ |y| \ge c\}$, i. e., over each point $z \in \Pi_c$ exactly p points of the sets $\{\mathcal{D}\}$ are lying with regard for their multiplicities; finally, none among the sets of the collection $\{\mathcal{D}\}$ contains a Jordan arc which covers entirely the straight line $\operatorname{Re} z = x$ for a certain x, $0 < x < 1$. Further, the family of closed sets $\{\mathcal{E}\}$ is called an admissible family corresponding to the family $\{\mathcal{D}\}$ if each set $\mathcal{E}$ from $\{\mathcal{E}\}$ belongs to a certain set $\mathcal{D} \in \{\mathcal{D}\}$ and if the sets of the collection $\{\mathcal{E}\}$ also p-multiply cover the half-strips Π_c for a certain $c > 0$. For the sake of simplicity, it is convenient to assume that each set $\mathcal{D}$ of the family $\{\mathcal{D}\}$ contains a certain closed set $\mathcal{E} \equiv \mathcal{E}(\mathcal{D})$ of the family $\{\mathcal{E}\}$ (perhaps, empty). We denote by $l_c(x, A)$ the linear Lebesgue measure of the set of points from A which are lying over the segment $[x - ic,\ x + ic]$. By the result of a linearly averaging transformation $\mathcal{L}^p$ of an admissible family of open sets $\{\mathcal{D}\}$ we call the set $\mathcal{L}^p\{\mathcal{D}\} = \{z = x + iy \in \Pi : y > c - \frac{1}{2p} \sum_{\{\mathcal{D}\}} l_c(x, \mathcal{D})\}$, where c is from the definition of the family $\{\mathcal{D}\}$. One can easily see that $\mathcal{L}^p\{\mathcal{D}\}$ does not depend on a choice of the number $c > 0$. In

Supported by the Russian Foundation for Basic Research (grant 99-01-00443) and the Programme "Universities of Russia" (grant 991282).