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Solution of Boundary-Value Problems for a Degenerating Elliptic Equation of the Second Kind by the Method of Potentials

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Abstract—In this paper we study basic boundary value problems for one multidimensional degenerating elliptic equation of the second kind. Using the method of potentials we prove the unique solvability of the mentioned problems. We construct a fundamental solution and obtain an integral representation for the solution to the equation. Using this representation we study properties of solutions, in particular, the principle of maximum. We state the basic boundary value problems and prove their unique solvability. We introduce potentials of single and double layers and study their properties. With the help of these potentials we reduce the boundary value problems to the Fredholm integral equations of the second kind and prove their unique solvability.

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INTRODUCTION

Let E_p^+ be the half-space $x_p > 0$ of the p-dimensional Euclidean space of points $x = (x', x_p)$, $x' = (x_1, x_2, \dots, x_{p-1})$; denote by D a finite domain in E_p^+ bounded by the open part Γ_0 of the hyperplane $x_p = 0$ and a hypersurface Γ.

In this paper we study interior and exterior boundary-value problems E and NE for a degenerating elliptic equation in the form

$$L_m[U] = \sum_{j=1}^{p-1} \frac{\partial^2 U}{\partial x_j^2} + x_p^m \frac{\partial^2 U}{\partial x_p^2} = 0, \tag{1}$$

where $m > 4$, $p \geq 3$.

Mathematical modeling of physical processes often leads to boundary-value problems for degenerating elliptic equations. There is a significant amount of works devoted to studying such problems (see, for example, the review in [1]). Let us mention here only the closest to the topic of this paper works of M. V. Keldysh [2], A. V. Bitsadze [3], and K. B. Sabitov [4, 5].

M. V. Keldysh investigated the first boundary-value problem for the equation

$$\frac{\partial^2 U}{\partial x^2} + y^m \frac{\partial^2 U}{\partial y^2} + a(x,y)\frac{\partial U}{\partial x} + b(x,y)\frac{\partial U}{\partial y} + c(x,y)U = 0. \tag{2}$$

In particular, he proved that if $m \geq 2$ and $b(x,0) \geq 0$, then the Dirichlet problem does not necessarily have a solution, but problem E always has a unique bounded solution in the domain D with $p = 2$ and continuous data for Γ.

In [3] A. V. Bitsadze replaced the boundedness condition with the weight boundary condition $\lim\limits_{y\to 0} \psi(x,y)U(x,y) = \varphi(x)$, where $\psi(x,y)$ is a known function such that $\lim\limits_{y\to 0} \psi(x,y) = 0$, and $\varphi(x)$ is a given continuous function.

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