On Rellich's Inequalities in Euclidean Spaces

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Received March 5, 2018

Abstract—On domains of Euclidean spaces we consider inequalities for test functions and their Laplacians. We describe a family of domains having vanishing Rellich constants. For the Euclidean space of dimension 4 we present a new version of the Rellich inequality. In addition, we prove new one-dimensional Rellich-type integral inequalities for linear combinations of test functions and their derivatives of orders one and two.

Introduction. In books [1] and [2] one can find basic results on the Rellich inequalities in Euclidean spaces. We will need the following F. Rellich's inequality [1] with sharp constants:

$$\int_{\mathbb{R}^n} |\Delta f|^2 \, dx \geq \frac{n^2(n-4)^2}{16} \int_{\mathbb{R}^n} \frac{|f|^2}{|x|^4} dx \quad \forall f \in C_0^\infty(\mathbb{R}^n \setminus \{0\}), \tag{1}$$

where $n \in \mathbb{N}$, $n \neq 2$. There are several generalizations of this inequality (see [3−8] and the bibliography therein). The known results are connected with some direct analogs of (1), when one compares integrals for functions of the type $|x|^{\alpha+4}|\Delta f|^2$ and $|x|^\alpha |f|^2$, as well as there are results when the Laplace operator is replaced by polyharmonic operators. Consider now an analog of inequality (1) for smooth test functions $f : \Omega \to \mathbb{R}$, defined in the domain $\Omega \subset \mathbb{R}^n$, i.e., in an open connected subset of the Euclidean space. Let $\Omega \neq \mathbb{R}^n$ and let $\partial\Omega$ be the boundary of the domain Ω. Then for every $x \in \Omega$ it is well-defined the distance $\text{dist}(x, \partial\Omega) = \inf_{y \in \partial\Omega} |x - y|$. Suppose that $\mu \in \mathbb{R}$. We will consider the inequality

$$\int_\Omega \frac{|\Delta f|^2 \, dx}{\text{dist}^{2\mu}(x, \partial\Omega)} \geq C_{2\mu}(\Omega) \int_\Omega \frac{|f|^2 \, dx}{\text{dist}^{4+2\mu}(x, \partial\Omega)} \quad \forall f \in C_0^\infty(\Omega), \tag{2}$$

where the constant $C_{2\mu}(\Omega)$ is defined as the maximum possible constant at this place, i.e.,

$$C_{2\mu}(\Omega) = \inf_{f \in C_0^\infty(\Omega), \, f \not\equiv 0} \frac{\int_\Omega |\Delta f|^2 \text{dist}^{-2\mu}(x, \partial\Omega) \, dx}{\int_\Omega |f|^2 \text{dist}^{-4-2\mu}(x, \partial\Omega) \, dx}.$$

Since $\text{dist}(x, \partial(\mathbb{R}^n \setminus \{0\})) = |x|$ and the constant in Rellich's inequality (1) is sharp, one has that $C_0(\mathbb{R}^n \setminus \{0\}) = n^2(n-4)^2/16$ for $n \in \mathbb{N} \setminus \{2\}$. The sharp constant $C_{2\mu}((\mathbb{R}^n \setminus \{0\})$ for all values of $\mu \in \mathbb{R}$ is defined by the following formula of Caldiroli and Musina [3]:

$$C_{2\mu}(\mathbb{R}^n \setminus \{0\}) = \min_{k \in \mathbb{N} \cup \{0\}} \left| \frac{(n-2)^2}{4} - (\mu+1)^2 + k(n-2+k) \right|^2. \tag{3}$$

Main results. From the definition of the constant $C_{2\mu}(\Omega)$ it follows that $C_{2\mu}(\Omega) \in [0, \infty)$ for every domain Ω and every value of $\mu \in \mathbb{R}$.

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