Scalar Field Cosmology

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Friedmann Cosmology

We consider the model of a selfgravitating scalar field ϕ with the potential of self-interaction $V(\phi)$. The action of such model is

$$S = \int d^4x\sqrt{-g}\left(\frac{R+\Lambda}{2\kappa} - \frac{1}{2}\phi_{,\mu}\phi_{,\nu}g^{\mu\nu} - V(\phi)\right). \qquad (1)$$

By standard way one can obtain the energy-momentum tensor (EMT)

$$T^{(sf)}_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\left(\frac{1}{2}\phi_{,\rho}\phi^{,\rho} + V(\phi)\right). \qquad (2)$$

Friedmann Cosmology

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-\epsilon r^2} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right), \qquad (3)$$

where $\epsilon = 0,\ \epsilon = 1,\ \epsilon = -1$ for the specially-flat, closed and open universe respectively. Most attention will be paid to the case of a specially-flat universe ($\epsilon = 0$).

Friedmann Cosmology

The Einstein equation and the equation of a scalar field dynamics in FRW metric (3) lead to the system of the scalar cosmology equations

$$\frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2} + \frac{2\epsilon}{a^2} = \kappa V(\phi), \tag{4}$$

$$-\frac{3\ddot{a}}{a} = \kappa\left(\dot{\phi}^2 - V(\phi)\right), \tag{5}$$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + V'(\phi) = 0. \tag{6}$$

Friedmann Cosmology

The equations (4) and (5) can, by equivalent way, be replaced by its sum and the linear combination 3(4)+(5). Including the Hubble parameter $H = \frac{\dot{a}}{a}$ the scalar cosmology equations can be rewrote in the form

$$H^2 + \frac{\epsilon}{a^2} = \frac{\kappa}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \tag{7}$$

$$\dot{H} - \frac{\epsilon}{a^2} = -\kappa\frac{1}{2}\dot{\phi}^2 \tag{8}$$

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \tag{9}$$

Given presentation has some advantage when one derives any equations from (7)-(9) using the rest two ones and the differential consequences of them.

Friedmann Cosmology

Another presentation of the scalar cosmology equations has been first time proposed by G. Ivanov in 1981. Suggesting the dependence of Hubble parameter H on a scalar field ϕ he made the transformation of the equations (7)-(9) for the spatially-flat universe to the form, which was called later as Hamilton-Jacobi-like form. The equation (8) is transformed to

$$\frac{dH}{d\phi} \equiv H' = -\kappa \frac{1}{2} \dot{\phi}. \tag{10}$$

Squared given equation and making the substitution $\frac{1}{2}\dot{\phi}^2$, expressed in term of H'^2, into (7) one can obtain the equation

$$\frac{2}{3\kappa} \left[\frac{dH}{d\phi} \right]^2 - H^2 = -\frac{\kappa}{3} V(\phi) \tag{11}$$

The special form of the potential

The structure of equation (11) prompts us the form of the potential which can give the exact solution. Indeed, let the potential $V(\phi)$ has the form

$$V(\phi) = -\frac{2}{3\kappa}F'^2 + (F(\phi) + F_*)^2 \qquad (12)$$

where $F = F(\phi)$ being any function on ϕ and $F_* = const$. Then the solution of equation (11) may be expressed as

$$H(\phi) = \sqrt{\frac{\kappa}{3}}(F(\phi) + F_*). \qquad (13)$$

Thus, one can directly from (11) obtain the potential if the Hubble parameter is given. And vice versa, if one sets the potential in the form (12), then the solution of (11) will be defined by the equation (13).

The potential in the polynomial's form

As an example let us choose the function $F(\phi)$ as the finite series on degrees of the field ϕ

$$F(\phi) = \sum_{k=0}^{p} \lambda_k \phi^k + F_*. \tag{14}$$

Under this circumstance the potential $V(\phi)$ takes the following form

$$V(\phi) = -\frac{2}{3\kappa}\left[\sum_{k=0}^{p-1}(k+1)\lambda_{k+1}\phi^k\right]^2 + \left[\sum_{k=0}^{p}\lambda_k\phi^k + F_*\right]^2. \tag{15}$$

The potential in the polynomial's form

Let us consider the simple case when $F_* = 0,\ k = 0,\ p = 1$. Then the potential becomes

$$V(\phi) = -\frac{2}{3\kappa}\lambda_1^2 + \lambda_0^2 + 2\lambda_0\lambda_1\phi + \lambda_1^2\phi^2. \tag{16}$$

The function $F(\phi)$ and the Hubble parameter are

$$F(\phi) = \lambda_0 + \lambda_1\phi, \quad H(\phi) = \sqrt{\frac{\kappa}{3}}(\lambda_0 + \lambda_1\phi). \tag{17}$$

If we set $\lambda_0 = 0$ and $\lambda_1^2 = m^2/2$ then we came to the solution for the massive scalar field, obtained by Ivanov in 1981 for the potential

$$V(\phi) = \frac{m^2\phi^2}{2} - \frac{m^2}{3\kappa}. \tag{18}$$

The potential in the polynomial's form

The solution is

$$\phi(t) = -m\sqrt{\frac{2}{3\kappa}}t + \phi_s = -m\sqrt{\frac{2}{3\kappa}}(t - t_*), \quad \phi_s = m\sqrt{\frac{2}{3\kappa}}t_*. \quad (19)$$

Index "s" ("singularity") is related to the values at the initial time $t = 0$. The Hubble parameter

$$H = m\sqrt{\frac{\kappa}{6}}\phi \quad (20)$$

leads to the scale factor

$$a = a_s \exp\left(-\frac{m^2}{6}t^2 + m\sqrt{\frac{\kappa}{6}}\phi_s t\right). \quad (21)$$

It is interesting to note that the same solution and its application for calculation of e-folds number was found later in 2001 by W-F. Wang.

The potential in the polynomial's form

When $\lambda_0 \neq 0$ the solution for the scale factor will differ by the factor a_s in front of the exponent

$$a(t) = \tilde{a}_s \exp\left(-\frac{m^2}{6}t^2 + m\sqrt{\frac{\kappa}{6}}\phi_s t\right). \quad (22)$$

Here $\tilde{a}_s = a_s \exp\left(\lambda_0\sqrt{\frac{\kappa}{3}}\right)$. The potential $V(\phi)$ then takes the form

$$V(\phi) = -\frac{2}{3\kappa}\lambda_1^2 + \lambda_0^2 + 2\lambda_0\lambda_1\phi + \lambda_1^2\phi^2 = \left(\frac{m\phi}{\sqrt{2}} + \lambda_0\right)^2 - \frac{m^2}{3\kappa}. \quad (23)$$

Let us note that the linear transformation of the field value without changing of the mass

$$\tilde{\phi} = \frac{\phi}{\sqrt{2}} + \frac{\lambda_0}{m} \quad (24)$$

leads to the potential (18) for the field $\tilde{\phi}$.

The potential in the polynomial's form

Let us consider the case when $k = 1, 2$. The function $F(\phi)$ takes the view

$$F(\phi) = \lambda_1 \phi + \lambda_2 \phi^2. \tag{25}$$

The scale factor is

$$a(t) = a_s \exp\left[-\sqrt{\frac{\kappa}{3}} \frac{\lambda_1^2}{4\lambda_2} t - \frac{\sqrt{3\kappa}}{8\lambda_2} \exp\left(-\frac{8\lambda_2}{\sqrt{3\kappa}} (t - t_*) \right) \right]. \tag{26}$$

To make comparison with Ivanov's results (in his notation) let us display the relations between parameters of the model

$$\lambda_1 = 0, \quad \mu = -\frac{16}{3\kappa} \lambda_2^2, \quad \lambda = -4\lambda_2^2, \quad 3\kappa\mu = 4\lambda.$$

Trigonometric potential

To obtain trigonometric potential we set the function $F(\phi)$ by the following way

$$F(\phi) = A\sin(\lambda\phi), \quad A, \ \lambda - const. \tag{27}$$

Then the potential takes the view

$$V(\phi) = -A^2\cos^2(\lambda\phi)\left(1 + \frac{2A^2\lambda^2}{3\kappa}\right) + A^2 \tag{28}$$

The solution for a scale factor is

$$a(t) = a_s\left[\cosh(A\lambda\sqrt{2}(t - t_*))\right]^{1/3} \tag{29}$$

To obtain Ivanov's solution, we set the parameter $\lambda : \ \lambda^2 = \frac{3\kappa}{2}$. Then the potential takes the form

$$V(\phi) = -A^2\cos\left(\sqrt{6\kappa}\phi\right). \tag{30}$$

Exponential potential

If we set

$$F(\phi) = A\exp(\mu\phi) \tag{31}$$

then the potential takes the form

$$V(\phi) = A^2\left(1 - \frac{2\mu^2}{3\kappa}\right)\exp(2\mu\phi). \tag{32}$$

In accordance with general procedure explained above one can obtain

$$H(\phi) = \sqrt{\frac{\kappa}{3}}A\exp(\mu\phi). \tag{33}$$

Then the dependence of scalar field on time t has logarithmic character

$$\phi(t) = -\frac{1}{\mu}\ln\left(\frac{2A\mu^2}{\sqrt{3\kappa}}t\right) + \phi_s. \tag{34}$$

Exponential potential

Scale factor is evaluated by power law

$$a(t) = a_s (t - t_*)^{\kappa/2\mu^2}. \tag{35}$$

Addition of the constant F_* to $F(\phi)$ leads to the generalization of the solution (35)

$$a(t) = a_s e^{H_* t} (t - t_*)^{\kappa/2\mu^2}, \quad H_* = \sqrt{\frac{\kappa}{3}} F_*. \tag{36}$$

Then the potential acquires the additional terms

$$V(\phi) = A^2 \left(1 - \frac{2\mu^2}{3\kappa}\right) \exp(2\mu\phi) + 2AF_* e^{\mu\phi} + F_*^2. \tag{37}$$

Exponential potential

Muslimov in 1990 found generalization of Ivanov's solution for the exponential potential. Let us represent the solution, which contain both ones. For the function $F(\phi)$ in the form (31) with the potential (37) we obtain

$$H(\phi)=\sqrt{\frac{\kappa}{3}}\left(Ae^{\mu\phi}+F_*\right). \tag{38}$$

$$\phi(t)=-\frac{1}{\mu}\ln\left(\frac{2A\mu^2}{\sqrt{3\kappa}}t+v_*\right) \tag{39}$$

To obtain Ivaniv's solution (in his notations) we set $v_*=0$ and take into account the relations

$$\alpha=A^2\left(1-\frac{2\mu^2}{3\kappa}\right),\quad \beta=2\mu,\quad A=\sqrt{\frac{\alpha}{1-\frac{\beta^2}{6\kappa}}} \tag{40}$$

Exponential potential

To obtain Muslimov's solution (in his notation) we set $v_* = 1$ and take into account the relations below

$$F_* = 0, \quad \Lambda = A^2\left(1 - \frac{2\mu^2}{3\kappa}\right), \quad A = 2/\mu, \quad B = A/\sqrt{3}$$

In the range of the method described above Muslimov in 1990 suggested new original approach for solving the scalar field cosmology equation.

The solution with inverse potential

The potential in the Muslimov's work is presented by the following way

$$V(\phi) = m^2\phi^{-\beta}\left(1 - \frac{1}{6}\beta^2\phi^{-2}\right), \quad \beta > 0. \tag{41}$$

It is not difficult to check that the same potential may be obtained from F function

$$F(\phi) = m\phi^{-\beta/2}. \tag{42}$$

Therefore it is clear that the solution can be obtained by the general scheme. The Hubble parameter is

$$H(\phi) = \sqrt{\frac{\kappa}{3}}m\phi^{-\beta/2} + H_* \tag{43}$$

As we know the influence of H_* on the result, for the sake of simplicity we may put $H_* = 0$.

The solution with inverse potential

Integrating (10) we can obtain

$$\phi(t) = [K_1(t-t_*)]^{2/(\beta+4)} + \phi_*, \quad K_1 = \sqrt{\frac{\kappa}{3}}\left(\frac{\beta+4}{2m\beta}\right). \qquad (44)$$

This result leads to the dependence of Hubble parameter on time

$$H(t) = \sqrt{\frac{\kappa}{3}} m \left[K_1(t-t_*)\right]^{-\beta/(\beta+4)}. \qquad (45)$$

Then the scale factor is

$$a = a_s \sqrt{\frac{\kappa}{3}} m \frac{\beta+4}{4} K_1^{-\beta/(\beta+4)} \exp\left(t^{4/(\beta+4)}\right). \qquad (46)$$

The solution of such type can be confronted with both very early and later time universe.

The solution with intermediate (hyperbolic) function

To simplify calculations, let us, following by Muslimov's work, introduce new variable

$$x = \sqrt{\frac{3\kappa}{2}}\phi \tag{47}$$

and the potential's function

$$f^2 = \frac{\kappa}{3}\left|V(\phi)\right|. \tag{48}$$

For this notations the Ivanov-Salopek-Bond equation (11) reduced to

$$\left(H'_x\right)^2 - H^2 = \mp f^2 \tag{49}$$

The upper case corresponds to the positive sign of the potential. (Mitrinovich, 1937).

The solution with intermediate (hyperbolic) function

Let us search for the solution in the form

$$H(x) = f(x)\cosh\left(\coth^{-1} y(x)\right), \quad y > 1, \tag{50}$$

when we are taking the upper sign [1]. Some another choice is for the lower sign

$$H(x) = f(x)\sinh\left(\tanh^{-1} y(x)\right), \quad y < 1. \tag{51}$$

Using the formulae above for transition to the function $y(x)$, we obtain

$$\left[f' \cosh u + f \sinh u \ u'\right]^2 - \left[f(x)\cosh u\right]^2 = -f^2. \tag{52}$$

Here, for the sake of briefness, we introduced the function u by the following way

$$u(y) = \coth^{-1} y(x). \tag{53}$$

[1] In Muslimov's work by mistake included inverse hyperbolical tangents instead of inverse hyperbolical cotangents

The solution with intermediate (hyperbolic) function

We can shift the second term to the right hand side and, using the property of the hyperbolic function, take the root from left and right hand side of the equation (considering all values as positive). As the result we obtain the equation

$$f' \cosh u + f \sinh u u' = f \sinh u. \tag{54}$$

The transition to the function $y(x)$ is performing by inverse substitution (53) and by disclosing the derivative

$$u' = \frac{y'}{1 - y^2}. \tag{55}$$

The solution with intermediate (hyperbolic) function

Finally we came to the the following relation

$$\frac{f'}{f} = \tanh(\coth^{-1} y)\left(1 - \frac{y'}{1-y^2}\right). \tag{56}$$

After simple algebraic transformations we arrive to the Abel equation

$$y' = \frac{f'}{f}y^3 - y^2 - \frac{f'}{f}y + 1. \tag{57}$$

Repeated the same procedure for the lower case, we came once again to the equation (57)[2].

[2] In Muslimov's work by mistake obtained other signs for the second and the forth terms

The method of generating functions (Chervon and Fomin, 2016)

There are many new investigations in scalar field cosmology were devoted to exact solution construction. Some of them are based on the investigation of Ivanov-Solopec-Bond equation. Let us review the method of generating functions.
The equations of scalar field cosmology may be represented by the following way ($\kappa = 1$):

$$\dot{H} = V - 3H^2, \tag{58}$$

$$\ddot{\phi} + \sqrt{\frac{3}{2}}\sqrt{\dot{\phi}^2 + 2V(\phi)}\,\dot{\phi} + \frac{dV}{d\phi} = 0. \tag{59}$$

The equation (59) can be solved by using the generating functions, that depend on the scalar field and the scale factor.

The first class of generating functions

Defining the function $F = F(\phi)$ via Kruger & Norbury (2000)

$$\dot{\phi} \equiv \pm\sqrt{(F-1)V} \tag{60}$$

the equation (59) transforms to

$$\ddot{\phi} \pm \sqrt{\frac{3}{2}} V \sqrt{F^2 - 1} + V' = 0 \tag{61}$$

Equations (60) and (61) form a set of coupled equations which are equivalent to equation (59). They can be uncoupled by differentiating (60) with respect to time and substituting for $\ddot{\phi}$ in (61).

The first class of generating functions

Solving for $\ddot{\phi} = \frac{1}{2}\frac{d\dot{\phi}^2}{d\phi}$ from (60) we have

$$\ddot{\phi} = \frac{1}{2}\left[(F-1)V' + VF'\right] \tag{62}$$

(with $F' \equiv \frac{dF}{d\phi}$) and inserting this expression for $\ddot{\phi}$ into (61) leads to

$$(F+1)V' + VF' \pm \sqrt{6}V\sqrt{F^2-1} = 0 \tag{63}$$

It can be seen that if one chooses $F \equiv F(\phi)$, then equation (63) is always separable and the potential is given by

$$V = \beta \exp\left(-\int \frac{F' \pm \sqrt{6}\sqrt{F^2-1}}{F+1}d\phi\right) \tag{64}$$

with $F \equiv F(\phi)$ and β is a constant.

The first class of generating functions

This may be simplified to

$$V = B\exp\left(\mp\sqrt{6}\int\sqrt{\frac{F-1}{F+1}}d\phi\right) \tag{65}$$

where B is a constant.
If one chooses $F = const$, equation(65) gives

$$V = B\exp\left(\mp\sqrt{6}\sqrt{\frac{F-1}{F+1}}\phi\right) \tag{66}$$

with solutions from (60) of the form

$$\phi(t) = \pm\frac{1}{\sqrt{6}}\sqrt{\frac{F+1}{F-1}}\ln\left[\pm\frac{\sqrt{6}(F-1)\sqrt{B}}{\sqrt{F+1}}(t-C)\right] \tag{67}$$

$$a(t) \propto t^{\frac{1}{3}\left(\frac{F+1}{F-1}\right)} \tag{68}$$

The first class of generating functions

The another exact solution can be obtained from $F = \cosh(\lambda\phi)$. Then equation (65) gives

$$V(\phi) = C\,(1 + \cosh\lambda\phi)^{\mp(2\sqrt{6}/\lambda)-1} \tag{69}$$

$$\dot{\phi} = \pm\sqrt{C}\frac{\sinh\lambda\phi}{(1+\cosh\lambda\phi)^{1\pm g'}} \tag{70}$$

This equation is only consistent if all upper or all lower signs are taken, i.e. one should not mix upper and lower signs. Setting $\lambda = \sqrt{6}$ we may find the example of exact solution in the form

$$a(t) = a_0\left[\exp(2\lambda\sqrt{C}\;t) - 1\right]^{\frac{1}{3}} \tag{71}$$

The second class of generating functions

Harko et al (2014) suggested the generation method by defining new function $f(\phi)$ so that

$$\dot{\phi} = \sqrt{f(\phi)}; \quad 2V(\phi)\sinh^2 G(\phi) = \dot{\phi}^2,$$
$$\tilde{F}(\phi) = \sqrt{f(\phi)/2 + V(\phi)}$$

and changing the independent variable from t to ϕ. The equation (59) becomes

$$\frac{dG}{d\phi} + \frac{1}{2V}\frac{dV}{d\phi}\coth G + \sqrt{\frac{3}{2}} = 0, \tag{72}$$

where the function G can be obtained from the scalar field with the use of the equation

$$G(\phi) = \operatorname{arccosh}\sqrt{1 + \frac{\dot{\phi}^2}{2V(\phi)}} \tag{73}$$

The second class of generating functions

We consider the case in which the scalar field potential can be represented as a function of G in the form

$$\frac{1}{2V}\frac{dV}{d\phi} = \sqrt{\frac{3}{2}}\,\alpha_1\,\tanh G, \tag{74}$$

where α_1 is an arbitrary constant. With this choice, the evolution equation takes the simple form

$$\frac{dG}{d\phi} = \sqrt{\frac{3}{2}}\,(1+\alpha_1)\,, \tag{75}$$

with the general solution given by

$$G\,(\phi) = \sqrt{\frac{3}{2}}\,(1+\alpha_1)\,(\phi-\phi_0)\,, \tag{76}$$

where ϕ_0 is an arbitrary constant of integration.

The second class of generating functions

With the use of this form of G, we obtain the self-interaction potential of the scalar field and the scale factor

$$V(\phi) = V_0 \cosh^{\frac{2\alpha_1}{1+\alpha_1}} \left[\sqrt{\frac{3}{2}} (1+\alpha_1)(\phi - \phi_0) \right], \tag{77}$$

$$a = a_0 \sinh^{\frac{1}{3(1+\alpha_1)}} \left[\sqrt{\frac{3}{2}} (1+\alpha_1)(\phi - \phi_0) \right] \tag{78}$$

The second class of generating functions

Let the function G has the following form

$$G = \text{arccoth}\left(\sqrt{\frac{3}{2}}\frac{\phi}{\alpha_2}\right), \qquad \alpha_2 = \text{constant}. \tag{79}$$

Then the scalar field potential is given by

$$V(\phi) = V_0\left(\frac{\phi}{\alpha_2}\right)^{-2(\alpha_2+1)}\left[\frac{3}{2}\left(\frac{\phi}{\alpha_2}\right)^2 - 1\right], \tag{80}$$

where V_0 is an arbitrary constant of integration. The time dependence of the scalar field is given by a simple power law,

$$\frac{\phi(t)}{\alpha_2} = \left[\frac{\sqrt{2V_0}\,(\alpha_2+2)}{\alpha_2}\right]^{\frac{1}{\alpha_2+2}}(t-t_0)^{\frac{1}{\alpha_2+2}}. \tag{81}$$

The second class of generating functions

The scale factor has the exponential dependence on a scalar field and time

$$a = a_0 \exp\left(\frac{\phi^2}{4\alpha_2}\right) = a_0 \exp\left\{ \frac{1}{4\alpha_2} \left[\frac{(\alpha_2 + 2)\sqrt{2V_0}}{\alpha_2} \right]^{\frac{2}{\alpha_2+2}} (t - t_0)^{\frac{2}{\alpha_2+2}} \right\}, \tag{82}$$

with a_0 an arbitrary constant of integration.

The third class of generating functions

In the paper Schunck and Mielke (1994) the equations (58)-(59) are written in the following form

$$\dot{H} = V(H) - 3H^2 \tag{83}$$

$$\dot{\phi} = \pm\sqrt{2}\sqrt{3H^2 - V(H)}, \tag{84}$$

where $V(\phi) = V(\phi(t)) = V(\phi(t(H))) = V(H)$.
The potential as the function of the Hubble parameter $V = V(H)$ is defined as

$$V(H) = 3H^2 + g(H) \tag{85}$$

and, by the choice of the graceful exit function $g(H)$, exact solutions of the system (58)-(59) are generated.
For the power–law and intermediate inflation

$$g(H) = -AH^n, \tag{86}$$

where n is real and A a positive constant

The third class of generating functions

For $n = 0$, the following solutions were found:

$$H(t) = -(At + C_1) \tag{87}$$

$$a(t) = a_0 \exp\left(-\frac{1}{2A}(At + C_1)^2 + C_2\right) \tag{88}$$

$$\phi(t) = \pm\sqrt{\frac{2A}{\kappa}}(At + C_1 - C_3) \tag{89}$$

$$V(\phi) = 3\left(\sqrt{\frac{\phi}{2A}} + C_3\right)^2 - A \tag{90}$$

The third class of generating functions

For $n=1$

$$H(t)=C_1\exp(-At) \tag{91}$$

$$a(t)=a_0\exp\left(-\frac{C_1}{A}\exp(-At)+\frac{C_2}{A}\right) \tag{92}$$

$$\phi(t)=\pm\sqrt{\frac{8}{A}}\left[\sqrt{C_1}\exp\left(\frac{-At}{2}-C_3\right)\right] \tag{93}$$

$$V(\phi)=\frac{A}{8}e^{2C_3}\phi^2\left(\frac{3A}{8}e^{2C_3}\phi^2-A\right) \tag{94}$$

The third class of generating functions

For $n = 2$

$$H(t) = \frac{1}{At + C_1} \tag{95}$$

$$a(t) = a_0(C_2(At + C_1))^{1/A} \tag{96}$$

$$\phi(t) = \pm\sqrt{\frac{2}{A}} \ln\left(\frac{1}{C_3(At + C_1)}\right) \tag{97}$$

$$V(\phi) = (3 - A)C_3^2 \exp(\pm\sqrt{2A}\phi) \tag{98}$$

The third class of generating functions

For $n \neq 0, 1, 2$

$$H = (A(n-1)(t+C_1))^{1/(1-n)} \tag{99}$$

$$a(t) = a_0 \exp\left[(A(n-1))^{1/(1-n)} \frac{1-n}{2-n} (t+C_1)^{(2-n)/(1-n)} \right] \tag{100}$$

$$\phi(t) + C_3 = \sqrt{\frac{2}{A}} \frac{2}{2-n} \Big[A(n-1)(t+C_1) \Big]^{(2-n)/(2(1-n))} \tag{101}$$

$$V(\phi) = \sqrt{\frac{A}{8}} (2-n) (\phi + C_3)^{2/(2-n)} \times \left(3\frac{A}{8}(2-n)^2 (\phi+C_3)^2 - A\left(\frac{A}{8}\right)^{n/2} (2-n)^n (\phi+C_3)^n \right) \tag{102}$$

The third class of generating functions

For new inflation the generation function is considered as a polynomial in H up to second order

$$g(H) = \frac{1}{G}H^2 + \left(D - \frac{2A}{G}\right) H + \frac{A^2}{G} - AD, \qquad (103)$$

where A, D, G are constants.
The new class of exact solutions were obtained by this representation.

The fourth class of generating functions

In the article (H-C. Kim, 2012) the Hubble parameter is presented as

$$H(\phi,\dot{\phi}) = -\frac{1}{3\dot{\phi}}\frac{dG^2(\phi)}{d\phi}, \tag{104}$$

where $G(\phi)$ – the generating function.

$$V(\phi) = G^2(\phi) - \frac{2}{3}[G'(\phi)]^2 \tag{105}$$

$$\dot{\phi} = -\frac{2}{\sqrt{3}}G'(\phi), \qquad H = \frac{\dot{a}}{a} = \frac{1}{\sqrt{3}}G(\phi). \tag{106}$$

For the constant potential $V(\phi) = \Lambda > 0$ two generating functions are considered:

$$G(\phi) = \sqrt{\Lambda} \tag{107}$$

This gives de Sitter space-time with the scale factor expanding exponentially.

The fourth class of generating functions

The second generating function is given by

$$G(\phi) = \frac{e^{\sqrt{\frac{3}{2}}\phi} + \Lambda e^{-\sqrt{\frac{3}{2}}\phi}}{2} \tag{109}$$

The scalar field and the scale factor behave as

$$\phi = \sqrt{\frac{2}{3}} \log\left(\sqrt{3}H_I \tanh\left(\frac{3H_I}{2}t\right)\right), \tag{110}$$

$$a(t) = a_0 \sinh^{1/3}(3H_I t). \tag{111}$$

The fourth class of generating functions

For the exponential potential

$$V(\phi) = \Lambda e^{\sqrt{6}\beta\phi}. \tag{112}$$

generating function has the form

$$G(\phi) = \sqrt{\frac{\Lambda}{1-\beta^2}} e^{\sqrt{\frac{3}{2}}\beta\phi}, \tag{113}$$

where a real generating function of this form exists only when $|\beta| < 1$.

The fourth class of generating functions

The scalar field and scale factor corresponding to this is given by

$$\phi(t)=-\sqrt{\frac{2}{3}}\frac{1}{\beta}\log\left(1+\beta^2\sqrt{\frac{3\Lambda}{1-\beta^2}}t\right),$$
$$a(t)=a_0\left(1+\beta^2\sqrt{\frac{3\Lambda}{1-\beta^2}}t\right)^{\frac{1}{3\beta^2}} \qquad (114)$$

The fourth class of generating functions

For the power-like potentials

$$G(\phi) = \sqrt{\Lambda}\Big(1 + \frac{\mu}{n+1}|\phi|^{n+1}\Big) \tag{115}$$

$$V(\phi) = \Lambda\Big(1 + \frac{\mu}{n+1}|\phi|^{n+1}\Big)^2 - \frac{2}{3}\Lambda\mu^2\phi^{2n}. \tag{116}$$

$$\begin{aligned} \phi(t) &= (2(n-1)\mu H_I t)^{-\frac{1}{n-1}}, \\ a(t) &= a_0 \exp\left(H_I t - \frac{1}{2(n+1)}(2(n-1)\mu H_I t)^{-\frac{2}{n-1}}\right) \end{aligned} \tag{117}$$

with $\phi_0 = \phi(0) = 0$ and taking into account stability of solutions.

The fifth class of generating functions

In the paper (Chimento et al, 2012) the potential of the scalar field is considered as

$$V[\phi(a)] = \frac{F(a)}{a^6}, \tag{118}$$

where $F(a)$ – the generating function.
Klein-Gordon's equation is

$$\frac{1}{2}\dot{\phi}^2 + V(\phi) - \frac{6}{a^6}\int da \frac{F}{a} = \frac{C}{a^6} \tag{119}$$

where C is an arbitrary integration constant.

The fifth class of generating functions

Then, the problem has reduced to quadratures:

$$\Delta t = \sqrt{3} \int \frac{da}{a} \left[\frac{6}{a^6} \int da \frac{F}{a} + \frac{C}{a^6} \right]^{-1/2} \tag{120}$$

$$\Delta \phi = \sqrt{6} \int \frac{da}{a} \left[\frac{-F + 6 \int daF/a + C}{6 \int daF/a + C} \right]^{1/2}, \tag{121}$$

where $\Delta t \equiv t - t_0$, $\Delta \phi \equiv \phi - \phi_0$ and t_0, ϕ_0 are arbitrary integration constants.

The fifth class of generating functions

For the generating function

$$F(a) = Ba^s \left(b + a^s\right)^n , \tag{122}$$

where $B > 0$, $b > 0$, s and n are constants and $s(n+1) = 6$, taking $C = 0$, the potential is obtained as

$$V(\phi) = B \left[\cosh\left(\frac{s}{2\sqrt{6}}\Delta\phi\right)\right]^{2n} \tag{123}$$

This potential has a nonvanishing minimum at $\Delta\phi = 0$ for $s > 0$, which is equivalent to an effective cosmological constant. When $s < 0$, the origin becomes a maximum, and the potential vanishes exponentially for large ϕ.

Superpotential method in cosmology

Einstein and scalar fields equations:

$$H^2 = \frac{\kappa}{3}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right], \tag{124}$$

$$\dot{H} = -\frac{\kappa}{2}\dot{\phi}^2, \tag{125}$$

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0. \tag{126}$$

The superpotential as the potential of total energy:

$$W(\phi) = \frac{1}{2}U^2(\phi) + V(\phi), \quad U(\phi) = \dot{\phi}. \tag{127}$$

Superpotential method in cosmology

The consequence of (124), (126)

$$\sqrt{3\kappa}U^2W^{1/2} = -W'. \tag{128}$$

Integrating (128) by W we obtain the relation:

$$W = \frac{3\kappa}{4}\left(\int U(\phi)d\phi\right)^2. \tag{129}$$

Then, by suggesting that evolution of the scalar field is given one can determine the superpotential by solving the integral in the right hand side of (129) $\int \dot{\phi}^2 dt$. Obtaining W, one can find H from Friedman equation (124) by the following way

$$H = \frac{\kappa}{4}\sqrt{3\kappa}\left(\int U(\phi)d\phi\right). \tag{130}$$

The Superpotential of Phantom Field

Let us define the superpotential Υ as

$$\Upsilon(\phi) = V(\phi) - \frac{1}{2}U^2(\phi), \;\; U(\phi) = \dot{\phi}. \tag{131}$$

Then the phantom field equation takes the form:

$$3HU(\phi) = -\Upsilon'. \tag{132}$$

Hence one can obtain an expression of the superpotential in quadratures

$$\Upsilon = \frac{3\kappa}{4}\left(\int U(\phi)d\phi\right)^2 \tag{133}$$

The Superpotential of Phantom Field

The Friedman equation in terms of superpotential

$$H^2 = \frac{\kappa}{3}\Upsilon(\phi) \qquad (134)$$

can be reduced to

$$H = \frac{\kappa}{2}\int U(\phi)d\phi. \qquad (135)$$

Now one can formulate an algorithm for searching the exact solutions in view of the known scalar field evolution. Namely, assuming the dependence of the phantom field on cosmic time is known, one can determine the superpotential by evaluation of integral in right-hand side of (133) over time $\int \dot{\phi}^2\, dt$. With the known superpotential one can obtain H from the Friedman equation trough the same integral $\int \dot{\phi}^2\, dt$, and then the scale factor can be found.

Logarithmic evolution of phantom field

Let us suppose

$$\phi = A\ln(\lambda t). \tag{136}$$

Then the solutions have the form

$$\Upsilon = \frac{3}{4M_p^2}\left(C_1 - \frac{A^2}{t}\right)^2, \tag{137}$$

$$H = \frac{1}{2M_p^2}\left(C_1 - \frac{A^2}{t}\right), \quad a(t) = C_2 t^{-\frac{A^2}{2M_p^2}} e^{\frac{C_1}{2M_p^2}t}, \tag{138}$$

$$V(\phi) = \frac{3}{4M_p^2}(C_1 - A^2\lambda e^{-\phi/A})^2 + \frac{A^2\lambda^2}{2}e^{-2\phi/A}. \tag{139}$$

Considered evolutions of phantom field

- $\phi = At^l, \quad l \neq 0, \quad l \neq 1/2$ – power-law evolution,
- $\phi = At^l, \quad l \neq 0, \quad l = 1/2$ – power-law evolution,
- $\phi = Ae^{-\lambda t}$ – exponential evolution,
- $\phi = A \ln\left(\tanh[\lambda t]\right)$
- $\phi = A \ln\left(\tan[\lambda t]\right)$
- $\phi = A \sinh^{-1}[\lambda t]$
- $\phi = A \sin^{-1}[\lambda t]$

The Superpotential of Tachyon Field

The lagrangian of a tachyon field

$$\mathcal{L}_T = -V(\varphi)\sqrt{1 - \varphi_{,\mu}\varphi^{,\mu}}. \tag{140}$$

The action for self-gravitating tachyon reads:

$$S_{sgt} = \int d^4x\sqrt{-g}\left(\frac{R}{2\kappa} + \mathcal{L}_T\right), \tag{141}$$

The energy-momentum tensor is

$$T^{\mu}_{\nu} = \mathrm{diag}(\rho, -p, -p, -p) \tag{142}$$

where

$$\rho = \frac{V(\varphi)}{\sqrt{1-\dot{\varphi}^2}}, \quad p = -V(\varphi)\sqrt{1-\dot{\varphi}^2}. \tag{143}$$

The Superpotential of Tachyon Field

Varying the action (141) in respect of the tachyon field φ, we obtain the tachyon field equation

$$\frac{\ddot{\varphi}}{1-\dot{\varphi}^2}+3H\dot{\varphi}\pm\frac{1}{V(\varphi)}\frac{\partial V(\varphi)}{\partial\varphi}=0. \tag{144}$$

Then we rewrite Einstein equation

$$H^2=\frac{\kappa}{3}\frac{V(\varphi)}{\sqrt{1-\dot{\varphi}^2}}. \tag{145}$$

The Superpotential of Tachyon Field

In analogy with superpotential construction for phantom scalar field we define W as

$$W(\phi) = \ln \frac{V(\phi)}{\sqrt{1 - U^2(\phi)}}, \qquad U(\phi) = \dot{\phi}. \tag{146}$$

Tachyon field equation reads

$$3HU(\phi) = -W_{,\phi}. \tag{147}$$

The formula for the superpotential:

$$W(\phi) = \ln \frac{4M_p^2}{3(\int U(\phi) d\phi)^2}.$$

The Superpotential of Tachyon Field

We find from Friedman equation

$$H^2 = \frac{e^{W(\phi)}}{3M_p^2},$$

the Hubble parameter

$$H = \frac{2}{3\int U(\phi)d\phi}.$$

and the potential

$$V(\phi) = e^{W(\phi)}\sqrt{1 - U^2(\phi)}.$$

Now we can proceed with exact solution construction with given evolution of the scalar field.

Logarithmic evolution of tachyon

Let us suppose $\phi = A\ln(\lambda t)$. The solution is:

$$W = \ln \frac{4M_p^2}{3(C_1 - \frac{A^2}{t})^2}, \quad (148)$$

$$H = \frac{2}{3(C_1 - \frac{A^2}{t})}, \quad a(t) = C_2 e^{\frac{2}{3C_1}t}\left(A^2 - C_1 t\right)^{\frac{2A^2}{3C_1^2}}, \quad (149)$$

$$V(t) = \sqrt{1 - \frac{A^2}{t^2}}\frac{4M_p^2}{3(C_1 - \frac{A^2}{t})^2}, \quad (150)$$

$$V(\phi) = \sqrt{1 - (A\lambda e^{-\phi/A})^2}\frac{4M_p^2}{3(C_1 - A^2\lambda e^{-\phi/A})^2}. \quad (151)$$

Considered evolutions of phantom field

- $\varphi = At^l, \quad l \neq 0, \quad l \neq 1/2$ – power-law evolution,
- $\varphi = At^l, \quad l \neq 0, \quad l = 1/2$ – power-law evolution,
- $\varphi = Ae^{-\lambda t}$ – exponential evolution,
- $\varphi = A\ln\left(\tanh[\lambda t]\right)$
- $\varphi = A\ln\left(\tan[\lambda t]\right)$
- $\varphi = A\sinh^{-1}[\lambda t]$
- $\varphi = A\sin^{-1}[\lambda t]$

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