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Complete Indeterminacy of the Nevanlinna–Pick Problem in the Class $S[a, b]$

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INTRODUCTION

The class of analytic functions $S[a, b]$ was introduced by M. G. Krein in connection with the study of a moment problem on a compact interval ([1], p. 527). The truncated Nevanlinna–Pick problem in the matrix class $S[a, b]$ is considered in [2, 3].

In this paper we study the Nevanlinna–Pick problem in the class $S[a, b]$ with an infinite number of complex interpolation nodes, assuming that all truncated problems are completely indeterminate. We prove that a problem with an infinite number of interpolation nodes either becomes degenerate, or remains completely indeterminate. We consider the multiplicative structure of resolvent matrices of truncated interpolation problems. We introduce generalized Stieltjes parameters. In terms of the convergence of series of Stieltjes parameters we obtain a generalized Stieltjes criterion for the complete indeterminacy of the Nevanlinna–Pick interpolation problem. We propose a special normalization of resolvent matrices of a sequence of truncated interpolation problems which guarantees their convergence to a resolvent matrix of an infinite interpolation problem. In a completely indeterminate case the set of all solutions of the Nevanlinna–Pick problem with an infinite number of interpolation nodes is described by linear-fractional transform (31).

1. NEVANLINNA–PICK PROBLEM IN THE CLASS $S[a, b]$

Let real values $a < b$ and a natural number m be given. Denote $\mathbb{C}_- = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$, $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, $\mathbb{C}_\pm = \mathbb{C}_- \cup \mathbb{C}_+$. Let $\mathbb{C}^{m\times m}$ stand for the set of complex square matrices of the order m; let $\mathbb{C}^{m\times m}_\geq$ and $\mathbb{C}^{m\times m}_>$ stand, correspondingly, for the set of nonnegative matrices and that of positive ones. For nonnegative (positive) matrices we write $A \geq 0$ ($A > 0$). Let the symbols $I_m \in \mathbb{C}^{m\times m}$ and $0_m \in \mathbb{C}^{m\times m}$ denote the unit and zero matrices.

Definition 1. The symbol $S[a, b]$ stands for holomorphic matrix functions (m. f.) $s : \mathbb{C} \setminus [a, b] \to \mathbb{C}^{m\times m}$ such that

$$\frac{s(z) - s^*(z)}{z - \overline{z}} \geq 0 \ \ \forall z \in \mathbb{C}_\pm, \quad s(x) \geq 0 \ \ \forall x \in \mathbb{R} \setminus [a, b].$$

Hereinafter $s^*(z)$ is a short notation for $(s(z))^*$.

Let an infinite sequence of pairwise distinct complex values $\mathcal{Z}_\infty = \{z_j\}_{j=1}^\infty \subset \mathbb{C}_+$ and an infinite sequence of matrices $\{s_j\}_{j=1}^\infty \subset \mathbb{C}^{m\times m}$ be given. In the Nevanlinna–Pick problem one has to describe all m. f. $s \in S[a, b]$ such that

$$s(z_j) = s_j \ \ \forall j \in \mathbb{N}. \tag{1}$$

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