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CASIMIR ELEMENTS OF $\mathbb{Z}$-FORMS OF MODULAR LIE ALGEBRAS

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For the semisimple Lie algebras over a field of characteristic zero, any element of the center of the universal enveloping algebra is known to be a generalized Casimir element. For the finite-dimensional Lie algebras over a field of positive characteristic, in [1] a hypothesis was suggested that any nontrivial central element is a generalized Casimir element. L.P. Bedratyuk (see [2]) proved this statement under some restrictions imposed upon the ring of invariants of the symmetric algebra of a Lie algebra. Using his method, in this article we prove an analogous theorem for an arbitrary algebra.

Let G be an algebra over a field k, M a vector space over the same field. Assume that a homomorphism of vector spaces $\varphi : G \to \operatorname{End}(M)$ exists. Denote by $A_M(G)$ the associative algebra generated by the operators φ_x. Since M is a left $A_M(G)$-module, we will say that M is a left G-module. In particular, if $M = G$ and φ_x is the operator of the left multiplication, M will be called the adjoint G-module (by analogy with the Lie algebras).

Let us extend the operator of the left multiplication $L_x : y \to xy$, $x, y \in G$, to the corresponding differentiation D_x, $x \in G$, of the symmetric algebra $S(G)$ of the algebra G. Let us define the algebra of invariants $S(G)^G$ as the set of all elements of $S(G)$ annihilated by the operators D_x. In the case where $\operatorname{char} k = p > 0$, we denote $S_p(G) = \{y^p \mid y \in S(G)\}$. Obviously, $S_p(G) \subset S(G)^G$. Elements of $S_p(G)$ will be called trivial invariants.

Let D be the associative algebra generated by the differentiations D_x, while D_L be the corresponding Lie algebra. The algebra $S(G)$ can be treated as a left D-module (or D_L-module). Suppose that contragradient D_L-modules P and P^* in $S(G)$ exist, and let $\{u_i\}$ and $\{u_i^*\}$ be their dual bases. Then the element $\Delta = \sum\limits_i u_i u_i^*$ belongs to $S(G)^G$, we will call it a generalized Casimir element. In this case, we say that the invariant $\Delta = \Delta(P, P^*)$ is determined by the modules P and P^*.

Theorem. *Let z be a homogeneous invariant from $S(G)^G$, $\deg z > 1$, $\dim_k G < \infty$. Then*

1) *if $\operatorname{char} k = 0$, then z is a generalized Casimir element $z = \Delta(P, P^*)$ and P is a G-submodule of the adjoint module*;
2) *if $\operatorname{char} k = p > 0$ and $\deg z \not\equiv 0 \pmod p$, then $z = \Delta(P, P^*)$, $P \subseteq G$.*

We subdivide the proof of Theorem into several steps. Let $e_1, \dots, e_n$ be a basis of the algebra G, G' a copy of G, $h_1, \dots, h_n$ a basis in G'. Assume that the correspondence $\varphi(h_i) = e_i$, $i = 1, \dots, n$, determines an isomorphism $\varphi : G' \to G$. Denote by $\widetilde{S}$ the direct complement to $G' \otimes 1$ in the tensor product of G-modules $G' \underset{k}{\otimes} S(G)$, where the action of G on G' is defined by the relations $e_i \circ h_j = \sum\limits_{k=1}^n c_{ijk} h_k$, c_{ijk} are the structure constants of G. The action of G on $G' \underset{k}{\otimes} S(G)$ defined by

$$g \circ (g' \otimes s) = (g \circ g') \otimes s + g' \otimes (g \circ s),$$