# AN EXISTENCE THEOREM FOR THE CAUCHY PROBLEM ON THE LIGHT-CONE FOR THE VACUUM EINSTEIN EQUATIONS WITH NEAR-ROUND ANALYTIC DATA 

Y. Choquet-Bruhat, P.T. Chruściel, J.M. Martín-García


#### Abstract

A class of characteristic general relativistic initial data satisfying a near-roundness condition at the tip of a light-cone is introduced. It is shown that for any such analytic data there exists a corresponding solution of the vacuum Einstein equations defined in a future neighborhood of the vertex.


Key words: characteristic Cauchy problem on a light-cone, vacuum Einstein equations.

## Introduction

In the paper [1] we studied the Cauchy problem for the Einstein equations with data on a characteristic cone $C_{O}$. We used the tensorial splitting of the Ricci tensor of a Lorentzian metric $g$ on a manifold $V$ as the sum of a quasidiagonal hyperbolic system acting on $g$ and a linear first order operator acting on a vector $H$, called the wavegauge vector. The vector $H$ vanishes if $g$ is in wave gauge; that is, if the identity map is a wave map from $(V, g)$ onto $(V, \widehat{g})$, with $\widehat{g}$ being some given metric, which we have chosen to be Minkowski. The data needed for the reduced PDEs is the trace, which we denote by $\bar{g}$, of $g$ on $C_{O}$. However, because of the constraints, the intrinsic, geometric, data is a degenerate quadratic form $\widetilde{g}$ on $C_{O}$. Given $\widetilde{g}$, the trace $\bar{g}$ is determined through a hierarchical system of ordinary differential equations ${ }^{1}$ along the rays of $C_{O}$, deduced from the contraction of the Einstein tensor with a tangent to the rays, which we have written explicitly and solved. We have called these equations the wave map gauge constraints and shown that they are necessary and sufficient conditions for the solutions of the hyperbolic system to satisfy the full Einstein equations. We have also proved local geometric uniqueness of a solution $g$ of the vacuum Einstein equations inducing a given $\widetilde{g}$ (for details see [1]). Further references to previous works on the problem at hand can be found in [1].

Existence theorems known for quasilinear wave equations with data on a characteristic cone give also existence theorems for the Einstein equations, if the initial data is Minkowski in a neighbourhood of the vertex. For more general data problems arise due to the apparent discrepancy between the functional requirements on the characteristic data of the hyperbolic system and the properties of the solutions of the constraints, due to the singularity of the cone $C_{O}$ at its vertex $O$. The aim of this work is to make progress towards resolving this issue, and provide a sufficient condition for the validity of an existence theorem in a neighbourhood of $O$ under conditions alternative to the fast-decay conditions of [4]. More precisely, we prove that analytic initial data arising

[^0]from a metric satisfying (3.3), (3.4) together with the "near-roundness" condition of Definition 6.1 lead to a solution of the vacuum Einstein equations to the future of the light-cone.

## 1. Cauchy problem on a characteristic cone for quasilinear wave equations

The reduced Einstein equations in wave-map gauge and Minkowski target are a quasi-diagonal, quasi-linear second order system for a set $v$ of scalar functions $v^{I}$, $I=1, \ldots, N$, on $\mathbf{R}^{n+1}$ of the form

$$
\begin{equation*}
A^{\lambda \mu}(y, v) D_{\lambda \mu}^{2} v+f(y, v, D v)=0, \quad y=\left(y^{\lambda}\right) \in \mathbf{R}^{n+1}, \quad n \geq 2, \quad f=\left(f^{I}\right) \tag{1.1}
\end{equation*}
$$

If the target is the Minkowski metric and takes in the coordinates $y^{\alpha}$ the canonical form

$$
\begin{equation*}
\eta \equiv-\left(d y^{0}\right)^{2}+\sum_{i=1}^{n}\left(d y^{i}\right)^{2} \tag{1.2}
\end{equation*}
$$

then,

$$
\begin{equation*}
D v=\frac{\partial v^{I}}{\partial y^{\lambda}}, \quad D_{\lambda \mu}^{2} v=\frac{\partial^{2} v^{I}}{\partial y^{\lambda} \partial y^{\mu}}, \quad \lambda, \mu=0,1, \ldots, n \tag{1.3}
\end{equation*}
$$

We will underline components in these $y^{\alpha}$ coordinates.
In the case of the Einstein equations the functions $A^{\lambda \mu} \equiv g^{\lambda \mu}$ do not depend directly on $y$, they are analytic in $v$ in an open set $W \subset \mathbf{R}^{N}$. For $v \in W$ the quadratic form $g^{\lambda \mu}$ is of Lorentzian signature. The functions $f^{I}$ are analytic in $v \in W$ and $D v \in \mathbf{R}^{(n+1) N}$, they do not depend directly on $y$ in vacuum.

The characteristic cone $C_{O}$ of vertex $O$ for a Lorentzian metric $g$ is the set covered by future directed null geodesics issued from $O$. We choose coordinates $y^{\alpha}$ such that the coordinates of $O$ are $y^{\alpha}=0$ and the components $A^{\lambda \mu}(0,0)$ take the diagonal Minkowskian values, $(-1,1, \ldots, 1)$. If $v$ is $C^{1,1}$ in a neighbourhood $U$ of $O$ and takes its values in $W$ there is an eventually smaller neighbourhood of $O$, still denoted $U$, such that $C_{O} \cap U$ is an $n$ dimensional manifold, differentiable except at $O$, and there exist in $U$ coordinates $y:=\left(y^{\alpha}\right) \equiv\left(y^{0}, y^{i}, i=1, \ldots, n\right)$ in which $C_{O}$ is represented by the equation of a Minkowskian cone with vertex $O$,

$$
\begin{equation*}
C_{O}:=\left\{r-y^{0}=0\right\}, \quad r:=\left\{\sum\left(y^{i}\right)^{2}\right\}^{1 / 2} \tag{1.4}
\end{equation*}
$$

and the null rays of $C_{O}$ represented by the generators of the Minkowskian cone, i.e. tangent to the vector $\ell$ with components $\underline{\ell^{0}}=1, \underline{\ell^{i}}=r^{-1} y^{i}$. Inspired by this result and following previous authors we will set the Cauchy problem for the equations (1.1) on a characteristic cone as the search of a solution which takes given values on a manifold represented by an equation of the form (1.4), that is, a set $v$ such that

$$
\begin{equation*}
\bar{v}=\varphi \tag{1.5}
\end{equation*}
$$

where overlining means restriction to $C_{O}$. The function $\varphi$ takes its values in $W$ and is such that $\ell$ is a null vector for $\bar{A}$, i.e. when $\bar{A} \equiv \bar{g}$

$$
\begin{equation*}
\underline{\ell^{\mu} \ell^{\nu} \bar{g}_{\mu \nu}}=\underline{\bar{g}_{00}}+2 r^{-1} y^{i} \underline{\bar{g}_{0 i}}+r^{-2} y^{i} y^{j} \bar{g}_{i j}=0 \tag{1.6}
\end{equation*}
$$

We use the following notations:

$$
\begin{aligned}
C_{O}^{T} & :=C_{O} \cap\left\{0 \leq t:=y^{0} \leq T\right\} \\
Y_{O} & :=\left\{y^{0}>r\right\}, \quad \text { the interior of } C_{O} \\
Y_{O}^{T} & :=Y_{O} \cap\left\{0 \leq y^{0} \leq T\right\} .
\end{aligned}
$$

and we set

$$
\begin{array}{rlrl}
\Sigma_{\tau} & :=C_{O} \cap\left\{y^{0}=\tau\right\}, & & \text { diffeomorphic to } S^{n-1} \\
S_{\tau}:=Y_{O} \cap\left\{y^{0}=\tau\right\}, & & \text { diffeomorphic to the ball } B^{n-1}
\end{array}
$$

We recall the following theorem, which applies in particular to the reduced Einstein equations.

Theorem 1.1. Consider the problem (1.1), (1.5). Suppose that:

1. There is an open set $U \times W \subset \mathbf{R}^{n+1} \times \mathbf{R}^{N}, Y_{O}^{T} \subset U$ where the functions $g^{\lambda \mu}$ are smooth in $y$ and $v$. The function $f$ is smooth ${ }^{2}$ in $y \in U$ and $v \in W$ and in $D v \in \mathbf{R}^{(n+1) N}$.
2. For $(y, v) \in U \times W$ the quadratic form $g^{\lambda \mu}$ has Lorentzian signature; it takes the Minkowskian values for $y=0$ and $v=0$. It holds that $\varphi(O)=0$.

3a. The function $\varphi$ takes its values in $W$. The cone $C_{O}^{T}$ is null for the metric $g^{\lambda \mu}(y, \varphi)$.

3b. $\varphi$ is the trace on $C_{O}^{T}$ of a smooth function in $U$.
Then there is a number $0<T_{0} \leq T<+\infty$ such that the problem (1.1), (1.5) has one and only one solution $v$ in $Y_{O}^{T_{0}}$ which can be extended by continuity to a smooth function defined on a neighbourhood of the origin in $\mathbf{R}^{n+1}$.

If $\varphi$ is small enough in appropriate norms, then $T_{0}=T$.

## 2. Null adapted coordinates

It has been shown (see [1] and references therein) that the constraints are easier to solve in coordinates $x^{\alpha}$ adapted to the null structure of $C_{O}$, defined by

$$
\begin{equation*}
x^{0}=r-y^{0}, \quad x^{1}=r \quad \text { and } x^{A}=\mu^{A}\left(r^{-1} y^{i}\right), \tag{2.1}
\end{equation*}
$$

where $A=2, \ldots, n$, local coordinates on the sphere $S^{n-1}$, or angular polar coordinates. Conversely

$$
y^{0}=x^{1}-x^{0}, \quad y^{i}=r \Theta^{i}\left(x^{A}\right) \quad \text { with } \quad \sum_{i=1}^{n} \Theta^{i}\left(x^{A}\right)^{2}=1
$$

In the $x$ coordinates the Minkowski metric (1.2) reads

$$
\begin{equation*}
\eta \equiv-\left(d x^{0}\right)^{2}+2 d x^{0} d x^{1}+\left(x^{1}\right)^{2} s_{n-1} \tag{2.2}
\end{equation*}
$$

with

$$
s_{n-1}:=s_{A B} d x^{A} d x^{B}, \quad \text { the metric of the round sphere } S^{n-1} .
$$

Recall that in these coordinates the non zero Christoffel symbols of the Minkowski metric are, with $S_{B C}^{A}$, the Christoffel symbols of the metric $s$,

$$
\begin{equation*}
\hat{\Gamma}_{1 A}^{B} \equiv \frac{1}{x^{1}} \delta_{A}^{B}, \quad \hat{\Gamma}_{A C}^{B} \equiv S_{A C}^{B}, \quad \widehat{\Gamma}_{A B}^{0} \equiv-x^{1} s_{A B}, \quad \hat{\Gamma}_{A B}^{1} \equiv-x^{1} s_{A B} \tag{2.3}
\end{equation*}
$$

In the general case, the null geodesics issued from $O$ have still equation $x^{0}=0$, $x^{A}=$ constant, so that $\ell:=\frac{\partial}{\partial x^{1}}$ is tangent to those geodesics. The trace $\bar{g}$ on $C_{O}$ of the spacetime metric $g$ that we are going to construct is such that $\bar{g}_{11}=0$ and $\bar{g}_{1 A}=0$; we use the notation

$$
\begin{equation*}
\bar{g} \equiv \bar{g}_{00}\left(d x^{0}\right)^{2}+2 \nu_{0} d x^{0} d x^{1}+2 \nu_{A} d x^{0} d x^{A}+\bar{g}_{A B} d x^{A} d x^{B}, \tag{2.4}
\end{equation*}
$$

[^1]We emphasize that our assumption that $\bar{g}$ is given by (2.4) is no geometric restriction for a metric $g$ to have such a trace on a null cone $x^{0}=0$.

The Lorentzian metric $g$ induces on $C_{O}$ a degenerate quadratic form $\widetilde{g}$ which reads in coordinates $x^{1}, x^{A}$

$$
\begin{equation*}
\widetilde{g} \equiv \widetilde{g}_{A B} d x^{A} d x^{B}, \tag{2.5}
\end{equation*}
$$

i.e. $\widetilde{g}_{11} \equiv \widetilde{g}_{1 A} \equiv 0$ while $\widetilde{g}_{A B} d x^{A} d x^{B} \equiv \bar{g}_{A B} d x^{A} d x^{B}$ is an $x^{1}$-dependent Riemannian metric on $S^{n-1}$ induced on each $\Sigma_{t}$ by $\widetilde{g}$, we denote it by $\widetilde{g}_{\Sigma}$. While $\widetilde{g}$ is intrinsically defined, it is not so for $\bar{g}_{00}, \nu_{0}, \nu_{A}$, they are gauge-dependent quantities.

Note that $\widetilde{g}$ has a more complicated expression in coordinates $y^{i}$ on $C_{O}$. Since the inclusion mapping of $C_{O}$ in the coordinates $y^{\alpha}$ is $y^{0}=r$, hence $\frac{\partial y^{0}}{\partial y^{i}}=\frac{y^{i}}{r}$, it holds that

$$
\begin{equation*}
\widetilde{g} \equiv \underline{\tilde{g}_{i j}} d y^{i} d y^{j}, \quad \text { with } \underline{\tilde{g}_{i j}} \equiv r^{-2} y^{i} y^{j} \underline{\bar{g}_{00}}+r^{-1}\left(y^{j} \underline{\bar{g}_{0 i}}+y^{i} \underline{\bar{g}_{0 j}}\right)+\underline{\bar{g}_{i j}} . \tag{2.6}
\end{equation*}
$$

For Theorem 1.1, to apply to the wave-gauge reduced Einstein equations, the components of the initial data in the $y$ coordinates must be the trace on $C_{O}$ of smooth spacetime functions. The solution of the reduced equations satisfy the full Einstein equations if and only if these initial data satisfy the wave-map gauge constraints. We have constructed in [1] these data as solutions of ODE in adapted null $x$ coordinates, which are admissible coordinates for $\mathbf{R}^{n+1}$ only for $r>0$. The change of coordinates from $x$ to $y$, smooth for $r>0$, is recalled below; the components of a spacetime tensor $T$ in the coordinates $x$ are denoted by $T_{\alpha \beta}$, while in the coordinates $y$ they are denoted by $T_{\alpha \beta}$.

Lemma 2.1. It holds that:

$$
\begin{gathered}
T_{00} \equiv \underline{T_{00}}, \quad T_{11} \equiv \underline{T_{00}}+2 \frac{y^{i}}{r} \underline{T_{0 i}}+\frac{y^{i}}{r} \frac{y^{j}}{r} \underline{T_{i j}}, \quad T_{01} \equiv-\left(\underline{T_{00}}+\underline{T_{0 i}} \Theta^{i}\right), \\
T_{0 A} \equiv-r \frac{\partial \Theta^{i}}{\partial x^{A}} \underline{T_{0 i}}, \quad T_{1 A} \equiv r \frac{\partial \Theta^{i}}{\partial x^{A}}\left(\underline{T_{0 i}}+\Theta^{j} \underline{T_{i j}}\right), \quad T_{A B} \equiv \underline{T_{i j}} r^{2} \frac{\partial \Theta^{i}}{\partial x^{A}} \frac{\partial \Theta^{j}}{\partial x^{B}} .
\end{gathered}
$$

Conversely, if $T_{1 A} \equiv T_{11} \equiv 0$, then

$$
\begin{gathered}
\underline{T_{00}} \equiv T_{00}, \quad \underline{T_{0 i}} \equiv-\left(T_{00}+T_{01}\right) r^{-1} y^{i}-T_{0 A} \frac{\partial x^{A}}{\partial y^{i}}, \\
\underline{T_{i j}}=\left(T_{00}+2 T_{01}\right) r^{-2} y^{i} y^{j}+T_{0 A} r^{-1}\left(y^{i} \frac{\partial x^{A}}{\partial y^{j}}+y^{j} \frac{\partial x^{A}}{\partial y^{i}}\right)+T_{A B} \frac{\partial x^{A}}{\partial y^{i}} \frac{\partial x^{B}}{\partial y^{j}} .
\end{gathered}
$$

Hereinafter, we shall often abbreviate partial derivatives as follows

$$
\begin{gathered}
\partial_{0} \equiv \frac{\partial}{\partial x^{0}}, \quad \partial_{1} \equiv \frac{\partial}{\partial x^{1}}, \quad \partial_{A} \equiv \frac{\partial}{\partial x^{A}} \\
\underline{\partial_{0}} \equiv \frac{\partial}{\partial y^{0}}, \quad \underline{\partial_{i}} \equiv \frac{\partial}{\partial y^{i}} .
\end{gathered}
$$

## 3. Characteristic data

3.1. Basic (intrinsic) characteristic data. The basic data on a characteristic cone for the Einstein equation is a degenerate quadratic form. We will define this data as the degenerate quadratic form $\widetilde{C}$ induced by a given Lorentzian spacetime metric $C$ which admits this cone as a null cone. By $C_{O}$, as before, we denote the manifold $x^{0} \equiv r-y^{0}=0$. If we take the $y^{i}$ as coordinates on $C_{O}$, it holds that, see (2.6),

$$
\begin{equation*}
\widetilde{C} \equiv \underline{\widetilde{C}_{i j}} d y^{i} d y^{j} \quad \text { with } \quad \underline{\widetilde{C}_{i j}} \equiv \underline{\bar{C}_{i j}}+r^{-1}\left(y^{j} \underline{\bar{C}_{i 0}}+y^{i} \underline{\bar{C}_{j 0}}\right)+r^{-2} y^{i} y^{j} \underline{\bar{C}_{00}}, \tag{3.1}
\end{equation*}
$$

where $\underline{\bar{C}_{\alpha \beta}}$ are the components in the $y$ coordinates of the trace $\bar{C}$ of $C$ on $C_{O}$ (not to be mistaken with the induced quadratic form $\widetilde{C})$. Assuming that $C_{O}$ is a null cone for $C$ with generators $\underline{\bar{\ell}^{0}}=1, \underline{\bar{\ell}^{i}}=\frac{y^{i}}{r}$, (1.6) implies that the quadratic form $\widetilde{C}$ is degenerate,

$$
\begin{equation*}
\frac{y^{i}}{r} \frac{y^{j}}{r} \underline{\widetilde{C}_{i j}} \equiv \underline{\bar{C}_{00}}+2 \frac{y^{i}}{r} \underline{\bar{C}_{0 i}}+\frac{y^{i}}{r} \frac{y^{j}}{r} \underline{\bar{C}_{i j}}=0 . \tag{3.2}
\end{equation*}
$$

Lemma 3.1. Two spacetime metrics $C$ and $C^{\prime}$ with components linked by

$$
\underline{\bar{C}_{i j}^{\prime}}:=\underline{\bar{C}_{i j}}+r^{-1}\left(a_{i} y_{j}+a_{j} y_{i}\right)+r^{-2} \alpha y^{i} y^{j}, \quad \underline{\bar{C}_{i 0}}{ }^{\prime}:=\underline{\bar{C}_{i 0}}-a_{i}, \quad \underline{\bar{C}_{00}}{ }^{\prime}:=\underline{\bar{C}_{00}}-\alpha
$$


Proof. Elementary calculation using the identity written above.
In what follows, to simplify computations we will make the restrictive condition that

$$
\begin{equation*}
\underline{C_{0 i}}=0, \quad \underline{C_{00}}=-1, \quad \text { i.e. } \quad C:=-\left(d y^{0}\right)^{2}+\underline{C_{i j}} d y^{i} d y^{j} . \tag{3.3}
\end{equation*}
$$

The set $\left\{y^{0}=r\right\}$ is then a null cone for the metric $C$, with generator $\underline{\ell^{0}}=1, \underline{\ell^{i}}=\frac{y^{i}}{r}$, if and only if

$$
\begin{equation*}
y^{i} \underline{\bar{C}_{i j}}=y^{j} \tag{3.4}
\end{equation*}
$$

(compare [5]).
The general relation between components in coordinates $y$ and adapted null coordinates $x^{1}, x^{A}$ gives

$$
\widetilde{C} \equiv \widetilde{C}_{A B} d x^{A} d x^{B} \quad \text { with } \quad \widetilde{C}_{A B} \equiv \bar{C}_{A B}, \quad \widetilde{C}_{1 A}=\widetilde{C}_{11}=0
$$

with $\widetilde{C}_{A B}$ being the components of a $x^{1} \equiv r$-dependent Riemannian metric on the sphere $S^{n-1}$

$$
-\left(\underline{C_{00}}+\frac{y^{i}}{r} \underline{C_{0 i}}\right) \equiv C_{01}=1, \quad C_{0 A} \equiv-\frac{\partial y^{i}}{\partial x^{A}} \underline{C_{0 i}}=0, \quad \underline{C_{00}} \equiv C_{00}=-1
$$

This metric $C$ is also such that

$$
C^{00} \equiv C^{0 A} \equiv C^{1 A} \equiv 0, \quad C^{01} \equiv C^{11} \equiv 1
$$

while $\bar{C}^{A B}$ are the elements of the inverse of the positive definite quadratic form with components $\bar{C}_{A B}$.
3.2. Full characteristic data. We have seen in [1] that the trace $\bar{g}$ of a Lorentzian metric $g$ satisfying the reduced Einstein equations is a solution of the full Einstein equations if and only if it satisfies the wave map gauge constraints. These constraints $\mathcal{C}_{\alpha}=0$ are deduced in vacuum from the identity satisfied by the Einstein tensor $S$ :

$$
\bar{\ell}^{\beta} \bar{S}_{\alpha \beta} \equiv \mathcal{C}_{\alpha}+\mathcal{L}_{\alpha}
$$

where $\mathcal{L}_{\alpha}$ is linear and homogeneous in the wave gauge vector $H$ while $\mathcal{C}_{\alpha}$ depends only on $\bar{g}$ and its derivatives among $C_{O}$ and the given target $\widehat{g}$. Given $\widetilde{g}$, i.e. $\bar{g}_{A B} \equiv \bar{C}_{A B}$, $\bar{g}_{1 A}=\bar{g}_{11}=0$, the remaining components $\nu_{0} \equiv \bar{g}_{01}, \nu_{A} \equiv \bar{g}_{0 A}, \bar{g}_{00}$ are determined by the constraints and limit conditions at the vertex $O$ which can always be satisfied
by choice of coordinates (see [1]). The Cagnac - Dossa theorem applies to components in the $y$ coordinates. Lemma 2.1 gives

$$
\begin{gather*}
\underline{\bar{g}_{00}} \equiv \bar{g}_{00}, \quad \underline{\bar{g}_{0 i}} \equiv-\left(\bar{g}_{00}+\nu_{0}\right) r^{-1} y^{i}-\underline{\nu_{i}}, \quad \text { with } \quad \underline{\nu_{i}}:=\nu_{A} \frac{\partial x^{A}}{\partial y^{i}}  \tag{3.5}\\
\underline{\bar{g}_{i j}}=\left(\bar{g}_{00}+2 \nu_{0}\right) r^{-2} y^{i} y^{j}+r^{-1}\left(y^{i} \underline{\nu_{j}}+y^{j} \underline{\nu_{i}}\right)+\bar{g}_{A B} \frac{\partial x^{A}}{\partial y^{i}} \frac{\partial x^{B}}{\partial y^{j}}
\end{gather*}
$$

while, for the chosen metric $C$ and $\bar{g}_{A B} \equiv \bar{C}_{A B}$

$$
\overline{\underline{C}}_{i j} \equiv r^{-2} y^{i} y^{j}+\bar{g}_{A B} \frac{\partial x^{A}}{\partial y^{i}} \frac{\partial x^{B}}{\partial y^{j}}
$$

Therefore,

$$
\begin{equation*}
\underline{\bar{g}_{i j}}=\underline{\bar{C}_{i j}}+\left(\bar{g}_{00}+2 \nu_{0}-1\right) r^{-2} y^{i} y^{j}+r^{-1}\left(y^{i} \underline{\nu_{j}}+y^{j} \underline{\nu_{j}}\right) . \tag{3.6}
\end{equation*}
$$

## 4. Null second fundamental form

We have defined in I the null second fundamental form of $\left(C_{O}, \widetilde{g}\right)$ as the tensor $\chi$ on $C_{O}$ defined by the Lie derivative ${ }^{3}$ with respect to the vector $\ell$ of the degenerate quadratic form $\widetilde{g}$, namely in the coordinates $x^{1}, x^{A}$ :

$$
\begin{gather*}
\chi_{A B}:=\frac{1}{2}\left(\mathcal{L}_{\ell} \widetilde{g}\right)_{A B} \equiv \frac{1}{2} \partial_{1} \bar{g}_{A B}  \tag{4.1}\\
\chi_{A 1}:=\frac{1}{2}\left(\mathcal{L}_{\ell} \widetilde{g}\right)_{A 1}=0, \quad \chi_{11}:=\frac{1}{2}\left(\mathcal{L}_{\ell} \widetilde{g}\right)_{11}=0 . \tag{4.2}
\end{gather*}
$$

In view of the application of the Cagnac - Dossa theorem, we look for smooth extensions. We define a smooth spacetime vector field $L$, vanishing at $O$ and with trace colinear with $\ell=\frac{\partial}{\partial x^{1}}$ on $C_{O}$, by its components in the $x^{\alpha}$ and $y^{\alpha}$ coordinates respectively:

$$
L:=y^{\lambda} \frac{\partial}{\partial y^{\lambda}} \equiv x^{0} \frac{\partial}{\partial x^{0}}+x^{1} \frac{\partial}{\partial x^{1}}, \quad \text { hence } \quad \bar{L} \equiv x^{1} \ell \equiv r \ell .
$$

We assume that the metric $C$ is smooth in $U$, a neighbourhood of $O$ in $\mathbf{R}^{n+1}$, i.e. its components $\underline{C_{i j}}$ are of class $C^{m}$, with $m$ as large as necessary in the considered context, functions of the $y^{\alpha}$. We define a symmetric $C^{m-1} 2$-tensor $X$, identically zero in the case where $C \equiv \eta$, the Minkowski metric, by:

$$
\begin{equation*}
X:=\frac{1}{2} \mathcal{L}_{L} C-C . \tag{4.3}
\end{equation*}
$$

In $y$ coordinates one has ${ }^{4}$, using $y^{i} \underline{C_{i j}}=y^{j}$

$$
\begin{equation*}
\underline{X_{00}} \equiv \underline{X_{0 i}} \equiv 0, \quad \underline{X_{i j}} \equiv \frac{1}{2}\left\{y^{0} \underline{\partial_{0} C_{i j}}+y^{h} \underline{\partial_{h} C_{i j}}\right\} . \tag{4.4}
\end{equation*}
$$

with

$$
y^{i} \underline{\partial_{0} C_{i j}}=0,
$$

[^2][^3]and, using $\partial_{h} y^{i}=\delta_{h}^{i}$
\[

$$
\begin{equation*}
y^{i} y^{h} \underline{\partial_{h} C_{i j}}=y^{h} \underline{\partial_{h}}\left(y^{i} \underline{C_{i j}}\right)-y^{h} \underline{C_{i j}} \underline{\partial_{h}} y^{i}=0 \tag{4.5}
\end{equation*}
$$

\]

which imply

$$
\begin{equation*}
\underline{L^{i} X_{i j}} \equiv y^{i} \underline{X_{i j}}=0 \tag{4.6}
\end{equation*}
$$

In $x$ coordinates we find, using the values of the components $C_{0 \alpha}$ and $C_{1 \alpha}$ of the metric $C$, that the tensor $X$ obeys the key properties

$$
\begin{equation*}
X_{\mu 0}=0, \quad X_{\mu 1}=0 \tag{4.7}
\end{equation*}
$$

while

$$
\begin{equation*}
X_{A B} \equiv \frac{1}{2}\left(x^{0} \partial_{0} C_{A B}+x^{1} \partial_{1} C_{A B}\right)-C_{A B} \tag{4.8}
\end{equation*}
$$

Hence $X_{A B}$ reduces on the null cone $C_{O}$ to

$$
\begin{equation*}
\bar{X}_{A B} \equiv \frac{1}{2} x^{1} \partial_{1} \bar{C}_{A B}-\bar{C}_{A B} \equiv x^{1} \chi_{A B}-\bar{g}_{A B} \tag{4.9}
\end{equation*}
$$

By $X$ we still denote the mixed $C^{m-1}$ tensor on spacetime obtained from $X$ by lifting an index with the contravariant associate of $C$; its $y$ components are the $C^{m-1}$ functions

$$
2 \underline{X_{\alpha}^{\gamma}} \equiv \underline{C^{\gamma \beta}}\left\{y^{0} \underline{\partial_{0} C_{\alpha \beta}}+y^{i} \underline{\partial_{i} C_{\alpha \beta}}\right\},
$$

hence $C$ being given by (3.3),

$$
\begin{equation*}
\underline{X_{i}^{j}} \equiv \frac{1}{2} \underline{C^{j h}}\left\{y^{0} \underline{\partial_{0} C_{i h}}+y^{k} \underline{\partial_{k} C_{i h}}\right\}, \quad \underline{X_{i}^{0}} \equiv \underline{X_{0}^{j}} \equiv \underline{X_{0}^{0}} \equiv 0, \tag{4.10}
\end{equation*}
$$

and

$$
\begin{equation*}
\underline{X_{i}^{j} L_{j}} \equiv 0 \tag{4.11}
\end{equation*}
$$

where the index of $L$ has been lowered with the metric $C$, so that this is equivalent to (4.6). In $x$ coordinates $X_{A}^{C}$ are the only non-vanishing components of $X$. Their traces on $C_{O}$ are

$$
\bar{X}_{A}^{C} \equiv \frac{1}{2} x^{1} \bar{g}^{B C} \partial_{1} \bar{g}_{A B}-\delta_{A}^{C} \equiv x^{1} \chi_{A}^{C}-\delta_{A}^{C},
$$

hence

$$
\begin{equation*}
\chi_{A}^{C}:=\frac{1}{2} \bar{g}^{B C} \partial_{1} \overline{g_{A B}}=\frac{1}{x^{1}}\left(\bar{X}_{A}^{C}+\delta_{A}^{C}\right) \tag{4.12}
\end{equation*}
$$

and

$$
\begin{equation*}
\tau:=\frac{1}{2} \bar{g}^{A B} \partial_{1} \overline{g_{A B}}=\frac{\overline{\operatorname{tr} X}}{x^{1}}+\frac{n-1}{x^{1}} . \tag{4.13}
\end{equation*}
$$

The trace of the tensor $X$ is the $C^{m-1}$ function

$$
\begin{equation*}
\operatorname{tr} X \equiv \underline{X_{\alpha}^{\alpha}} \equiv X_{\lambda}^{\lambda} \equiv C^{A B} X_{A B} . \tag{4.14}
\end{equation*}
$$

On the light cone $C_{O}$ it holds that

$$
\begin{align*}
& \overline{\operatorname{tr} X} \equiv \underline{\bar{X}_{i}^{i}} \equiv \bar{g}^{A B} \bar{X}_{A B} \equiv \frac{x^{1}}{2} \bar{g}^{A B} \partial_{1} \bar{g}_{A B}-(n-1)  \tag{4.15}\\
& |\chi|^{2}:=\chi_{A}^{C} \chi_{C}^{A} \equiv \frac{1}{\left(x^{1}\right)^{2}}\left\{\bar{X}_{\beta}^{\alpha} \bar{X}_{\alpha}^{\beta}+2 \overline{\operatorname{tr} X}+n-1\right\} \tag{4.16}
\end{align*}
$$

## 5. A criterium: admissible series

To show that the integration of the constraints, which appear as ODE in $x^{1}$, leads to traces on the cone of smooth spacetime functions, we shall use the following lemma, introduced by Cagnac (unpublished) for formal series, but used here for real analytic functions, a special class $C^{\omega}$ of $C^{\infty}$ functions.

Lemma 5.1. A function is the trace $\bar{f}$ on $C_{O}^{T}$ of a spacetime function $f$ analytic in $U \cap Y_{O}^{T}, U$ is a neighbourhood of $O$, if and only if it admits on $U \cap C_{O}^{T}$ a convergent expansion of the form

$$
\begin{equation*}
\bar{f} \equiv f_{0}+\sum_{p=1}^{\infty} \bar{f}_{p} r^{p} \tag{5.1}
\end{equation*}
$$

with

$$
\begin{equation*}
\bar{f}_{p} \equiv \bar{f}_{p, i_{1} \ldots i_{p}} \Theta^{i_{1}} \ldots \Theta^{i_{p}}+\bar{f}_{p, i_{1} \ldots i_{p-1}}^{\prime} \Theta^{i_{1}} \ldots \Theta^{i_{p-1}} \tag{5.2}
\end{equation*}
$$

where $f_{0}, f_{p, i_{1} \ldots i_{p}}$ and $f_{p, i_{1} \ldots i_{p-1}}^{\prime}$ are numbers. Such a series is called an admissible series. A coefficient $f_{p}$ of the form (5.2) is called an admissible coefficient of order $p$.

Proof. If $f$ is analytic it admits an expansion in Taylor series

$$
\begin{equation*}
f \equiv \sum_{p=0}^{\infty} f_{\alpha_{1} \ldots \alpha_{p}} y^{\alpha_{1}} \ldots y^{\alpha_{p}}, \quad f_{\alpha_{1} \ldots \alpha_{p}}:=\frac{1}{p!} \frac{\partial^{p} f}{\partial y^{\alpha_{1}} \ldots \partial y^{\alpha_{p}}}(O) . \tag{5.3}
\end{equation*}
$$

One goes from the formulas (5.3) to (5.1), (5.2) by replacing $y^{i}$ by $r \Theta^{i}$ and $y^{0}$ by $r$, and conversely, in $\Omega \cap C_{O}$ or in $\Omega$.

Remark 5.1. The identity (5.1) is equivalent to saying that $\bar{f}$ is of the form $\bar{f}=f_{1}+r f_{2}$, with $f_{1}$ and $f_{2}$ analytic functions of $y^{i}$.

We say that an admissible series is of minimal order $q$ if the coefficients $f_{p}$ are identically zero for $p<q$.

Proposition 5.1. If the metric $C$ is analytic and satisfies the conditions (3.3), (3.4) then the functions $\overline{\operatorname{tr} X}$ and $|X|^{2}$ are admissible series of minimal orders 2 and 4 respectively.

The following lemmas will be very useful when integrating the constraints.
Lemma 5.2. If $f_{p}$ and $h_{q}$ are admissible coefficients of order $p$ and $q$ respectively, then $f_{p}+h_{p}$ and $f_{p} h_{q}$ are admissible coefficients of order $p$ and $p+q$ respectively.

Proof. Elementary computation of $\left(f_{p}+h_{p}\right) r^{p}$ and $f_{p} h_{q} r^{p+q}$ replacing $r \Theta^{i}$ by $y^{i}$ and $r^{2}$ by $\Sigma_{i}\left(y^{i}\right)^{2}$.

Suppose that $f$ and $h$ are admissible series of minimal orders $q_{f}$ and $q_{h}$. The following are easy-to-check consequences of the lemma:

1) $f h$ is an admissible series of minimal order $q_{f}+q_{h}$;
2) if $q_{f}=q_{h}$, then $f+h$ is an admissible series of the same minimal order;
3) if $f(0) \neq 0$ and $q_{f}=0$, then $1 / f$ is an admissible series of also minimal order 0 ;
4) $r \partial_{1} f$ is an admissible series of minimal order $q_{f}$, unless $q_{f}=0$ and then it has a larger minimal order.

Lemma 5.3. If $k$ and $h$ are admissible series with $h$ of minimal order $q_{h} \geq 1$ and the constant $k_{0} \equiv k(0) \geq 0$, then the $O D E$

$$
\begin{equation*}
r \partial_{1} f+k f=h \tag{5.4}
\end{equation*}
$$

admits one and only one solution $f$ which is also an admissible series of the same minimal order $q_{h}$ as $h$. The result extends to $q_{h}=0$ if $k_{0}>0$.

Proof. Expand

$$
\begin{equation*}
f=\sum_{p=0}^{\infty} f_{p} r^{p}, \quad k=\sum_{p=0}^{\infty} k_{p} r^{p}, \quad h=\sum_{p=q_{h}}^{\infty} h_{p} r^{p} \tag{5.5}
\end{equation*}
$$

hence

$$
r \partial_{1} f=\sum_{p=1}^{\infty} p f_{p} r^{p}
$$

plug into the ODE (5.4) and proceed to identifications.
We obtain by equating to zero the constant term

$$
\begin{equation*}
k_{0} f_{0}=h_{0} \tag{5.6}
\end{equation*}
$$

a relation which can be satisfied when $h_{0} \neq 0$ only when $k_{0} \neq 0$. We first consider the case where $h_{0}=0$, i.e. $q_{h} \geq 1$, and take $f_{0}=0$. We get the successive equalities

$$
\begin{equation*}
f_{1}+k_{0} f_{1}=h_{1}, \quad \text { i.e. } \quad f_{1}=\frac{h_{1}}{1+k_{0}}, \tag{5.7}
\end{equation*}
$$

and the recurrence relation, using $f_{0}=0$,

$$
\begin{equation*}
\left(p+k_{0}\right) f_{p}+\sum_{q=1}^{p-1} k_{q} f_{p-q}=h_{p} . \tag{5.8}
\end{equation*}
$$

For $p<q_{h}$, we have $h_{p}=0$ and the recurrence relation gives $f_{p}=0$. Therefore, the leading admissible coefficients of $f$ and $h$ are always related by

$$
\begin{equation*}
f_{q_{h}}=\frac{h_{q_{h}}}{q_{h}+k_{0}} . \tag{5.9}
\end{equation*}
$$

We assume the series for $k$ and $h$ converge for all directions $\Theta^{i}$ and radius $c r<1$; that is, we assume that there exists a constant $c$ such that

$$
\begin{equation*}
\left|k_{p}\right|<c^{p}, \quad \text { and } \quad\left|h_{p}\right|<c^{p} \tag{5.10}
\end{equation*}
$$

Since $k_{0} \geq 0$ we have

$$
\left|f_{1}\right| \leq\left|\frac{h_{1}}{1+k_{0}}\right|<\frac{c}{1+k_{0}} \leq c
$$

Assume now that

$$
\begin{equation*}
\left|f_{p}\right|<c^{p} \quad \text { for } \quad p<p_{0} \tag{5.11}
\end{equation*}
$$

then from the iteration we get, for larger values of $p$, the inequality

$$
\begin{equation*}
\left|f_{p}\right|<\frac{p}{p+k_{0}} c^{p} \leq c^{p} \tag{5.12}
\end{equation*}
$$

The bounds on $\left|f_{p}\right|$ show that the series for $f$ also converges. It is an admissible series of minimal order $q_{f}=q_{h}$.

When $q_{h}=0$, i.e. $h_{0} \neq 0$ and $k_{0} \neq 0$ we take

$$
f_{0}=\frac{h_{0}}{k_{0}}
$$

and we set

$$
F=f-f_{0}
$$

It satisfies the equation

$$
\begin{equation*}
r \partial_{1} F+k F=H, \quad \text { with } \quad H:=h-k f_{0} \tag{5.13}
\end{equation*}
$$

We have

$$
H_{0}=0
$$

and we apply the previous result to $F$.
Corollary 5.1. If $f$ and $h$ are admissible series related by (5.4) and $p+k_{0}>0$, and $r^{-p} h$ is an admissible series then $r^{-p} f$ is an admissible series of the same minimal order.

Proof. Set $f=r^{p} \phi$. If $f$ satisfies (5.4) then $\phi$ satisfies the equation

$$
r \partial_{1} \phi+(p+k) \phi=r^{-p} h
$$

Remark 5.2. The following example is a case of a differential equation of the form (5.4) with $q_{h}=1$, but $k_{0}$ is a negative integer, which does not admit as a solution an admissible series. Let

$$
k=\frac{1}{r-1}=-1-r-r^{2}-r^{3}-\cdots, \quad h=\frac{r}{1-r^{2}}=r+r^{3}+r^{5}+\cdots
$$

We can solve the ODE explicitly,

$$
f=\frac{r}{r-1}\left(f_{\infty}+\log \frac{r+1}{r}\right)
$$

with $f_{\infty}$ an arbitrary integration constant, which cannot be expanded in powers of $r$ near 0 . However if we change $k$ to $r /(r-1)$, then $k_{0}$ changes from -1 to 0 , the problem disappears. Remark that the problem also disappears if we change $h$ to $r^{2} /\left(1-r^{2}\right)$, i.e. $q_{h}=2$.

In what follows, we will assume the metric $C$, of the form (3.3) and satisfying (3.4) is analytic, takes Minkowskian values at the vertex $O$, and is such that the components of its trace on $C_{O}$ satisfy

$$
\begin{equation*}
\underline{\bar{C}_{i j}} \equiv \delta_{i j}+\underline{\bar{c}_{i j}}, \quad \underline{\bar{C}^{i j}} \equiv \delta^{i j}+\underline{\bar{c}^{i j}} \tag{5.14}
\end{equation*}
$$

where $\underline{\bar{c}_{i j}}$ and $\underline{\bar{c}^{i j}}$ have admissible expansions of minimal order 2 while $\overline{\partial_{0} c_{i h}}$ has an admissible expansion of minimal order 1. The definition (4.10) implies then that

$$
\begin{equation*}
\underline{\bar{X}_{i}^{j}} \equiv \frac{1}{2} \underline{C^{j h}}\left\{r \overline{\partial_{0} c_{i h}}+y^{k} \overline{\partial_{k} c_{i h}}\right\} \tag{5.15}
\end{equation*}
$$

has an admissible expansion of minimal order 2.

## 6. The first wave-map gauge constraint

We have deduced our first constraint(see [1]) from the identity

$$
\begin{equation*}
\bar{\ell}^{\beta} \bar{S}_{1 \beta} \equiv \bar{R}_{11} \equiv-\partial_{1} \tau+\nu^{0} \partial_{1} \nu_{0} \tau-\frac{1}{2} \tau\left(\bar{\Gamma}_{1}+\tau\right)-\chi_{A}^{B} \chi_{B}^{A} \tag{6.1}
\end{equation*}
$$

with

$$
\begin{equation*}
\bar{\Gamma}_{1} \equiv \bar{W}_{1}+\bar{H}_{1}, \quad \bar{W}_{1} \equiv-\nu_{0} \bar{g}^{A B} r s_{A B} . \tag{6.2}
\end{equation*}
$$

Hence for the first wave-map gauge constraint in vacuum we have the equation

$$
\begin{equation*}
\mathcal{C}_{1}:=-\partial_{1} \tau+\nu^{0} \partial_{1} \nu_{0} \tau-\frac{1}{2} \tau\left(\tau-\nu_{0} \bar{g}^{A B} r s_{A B}\right)-\chi_{A}^{B} \chi_{B}^{A}=0 . \tag{6.3}
\end{equation*}
$$

When $\bar{g}_{A B}$ is known this equation reads as a first order differential equation for $\nu_{0}$

$$
\begin{equation*}
\nu^{0} \partial_{1} \nu_{0}=\tau^{-1} \partial_{1} \tau+\frac{1}{2}\left(\tau-\nu_{0} \bar{g}^{A B} r s_{A B}\right)+\tau^{-1} \chi_{A}^{B} \chi_{B}^{A} \tag{6.4}
\end{equation*}
$$

It can be written as a linear equation for $\nu^{0}-1$,

$$
\begin{equation*}
\partial_{1} \nu^{0}+a\left(\nu^{0}-1\right)+b=0 \tag{6.5}
\end{equation*}
$$

with

$$
\begin{gather*}
a:=\tau^{-1} \partial_{1} \tau+\frac{1}{2} \tau+\tau^{-1}|\chi|^{2}, \quad|\chi|^{2} \equiv \chi_{A}^{B} \chi_{B}^{A},  \tag{6.6}\\
b:=a-\frac{1}{2} \bar{g}^{A B} r s_{A B} . \tag{6.7}
\end{gather*}
$$

In the flat case $\bar{g}^{A B}=\eta^{A B}, \tau=\frac{n-1}{r}, \chi_{A}^{B}=\frac{1}{r} \delta_{A}^{B}$ the equation reduces to:

$$
\partial_{1} \nu^{0}+\frac{1}{2}\left(\nu^{0}-1\right) \frac{n-1}{r}=0
$$

it has one solution tending to 1 when $r$ tends to zero, $\nu_{0}=1$. In the general case (6.5) reads, with $f:=\nu^{0}-1$,

$$
\begin{equation*}
r \partial_{1} f+k f+h=0, \quad k:=a r, \quad h:=b r=a r-\frac{1}{2} \bar{g}^{A B} r^{2} s_{A B} . \tag{6.8}
\end{equation*}
$$

Recall that $x^{1} \equiv r$ and

$$
\begin{equation*}
\chi_{A}^{C} \equiv \frac{1}{2} \bar{g}^{B C} \partial_{1} \overline{g_{A B}}=\frac{1}{r}\left(\bar{X}_{A}^{C}+\delta_{A}^{C}\right) . \tag{6.9}
\end{equation*}
$$

Hence

$$
\begin{gather*}
|\chi|^{2}=\frac{|\bar{X}|^{2}+2 \overline{\operatorname{tr} X}+n-1}{r^{2}} \\
\tau=\frac{\overline{\operatorname{tr} X}}{r}+\frac{n-1}{r}, \quad \tau^{-1}=\frac{r}{n-1+\overline{\operatorname{tr} X}} \tag{6.10}
\end{gather*}
$$

where $\overline{\operatorname{tr} X}$ is an admissible series of minimal order 2. The function $\left\{1+\frac{1}{n-1} \overline{\operatorname{tr} X}\right\}^{-1}$ is the trace of a $C^{\omega}$ function as long as $1+\frac{1}{n-1} \operatorname{tr} X$ does not vanish, hence always in a neighbourhood of $O$ since $\operatorname{tr} X$ vanishes there.

It holds that

$$
\begin{gather*}
\partial_{1} \tau \equiv-\frac{n-1+\overline{\operatorname{tr} X}}{r^{2}}+\frac{\partial_{1} \overline{\operatorname{tr} X}}{r}  \tag{6.11}\\
\tau^{-1} \partial_{1} \tau \equiv-\frac{1}{r}+\frac{\partial_{1} \overline{\operatorname{tr} X}}{n-1+\overline{\operatorname{tr} X}} \tag{6.12}
\end{gather*}
$$

Also we can write

$$
\begin{equation*}
\tau^{-1}|\chi|^{2} \equiv \frac{|\bar{X}|^{2}+2 \overline{\operatorname{tr} X}+n-1}{r(n-1+\overline{\operatorname{tr} X})} \equiv \frac{|\bar{X}|^{2}+\overline{\operatorname{tr} X}}{r(n-1+\overline{\operatorname{tr} X})}+\frac{1}{r} . \tag{6.13}
\end{equation*}
$$

Finally computation gives

$$
\begin{equation*}
k \equiv a r \equiv \frac{n-1}{2}+\frac{r \partial_{1} \overline{\operatorname{tr} X}+\overline{\operatorname{tr} X}+|\bar{X}|^{2}}{n-1+\overline{\operatorname{tr} X}} \tag{6.14}
\end{equation*}
$$

We see that in a neighbourhood of $O, k-\frac{n-1}{2}$ admits an admissible development of minimal order 2 .

On the other hand, since in the $x$ coordinates $\eta_{1 \alpha}=0, r^{2} s_{A B}=\eta_{A B}$ and we have assumed that $\bar{C}^{00}=\bar{C}^{0 A}=0, \bar{C}^{01}=1, \bar{g}^{A B}=\bar{C}^{A B}$, we have

$$
\bar{g}^{A B} r^{2} s_{A B} \equiv \bar{g}^{A B} \eta_{A B} \equiv \bar{C}^{A B} \eta_{A B} \equiv \bar{C}^{\alpha \beta} \eta_{\alpha \beta}-2
$$

Hence using now the values of $\underline{C_{\alpha \beta}}$

$$
\begin{equation*}
\frac{1}{2} \bar{g}^{A B} r^{2} s_{A B} \equiv \frac{1}{2} \overline{\left(1+C^{i j} \delta_{i j}-2\right)} \equiv \frac{n-1}{2}+\frac{1}{2} \overline{c^{i j} \delta_{i j}}, \tag{6.15}
\end{equation*}
$$

where $\overline{c^{i j} \delta_{i j}}$ has an admissible development of minimal order 2 . We conclude that

$$
\begin{equation*}
h \equiv \frac{r \partial_{1} \overline{\operatorname{tr} X}+\overline{\operatorname{tr} X}+|\bar{X}|^{2}}{(n-1+\overline{\operatorname{tr} X})}-\frac{1}{2} \frac{\overline{c^{i j} \delta_{i j}}}{(n} \tag{6.16}
\end{equation*}
$$

admits also such a development. Lemma 5.3 applies, and we have proved the following theorem:

Theorem 6.1. If the basic characteristic data are induced on $C_{O}$ by a $C^{\omega}$ (i.e. analytic) metric of the form (3.3), hence satisfying (2.4), then $\nu^{0}-1$ admits an admissible expansion of minimal order 2 , hence is the trace in a neighbourhood of $O$ of a $C^{\omega}$ spacetime function. Then $\nu^{0}=\bar{N}^{0}$ with $N^{0} \in C^{\omega}$ and $N^{0}(O)=1$. In a neighbourhood of $O$, it holds that $\nu_{0}=\bar{N}_{0}, N_{0}=\left(N^{0}\right)^{-1}$.

In the expression of the characteristic initial data in $y$ coordinates there appears $r^{-2}\left(\nu_{0}-1\right)$, which, though being continuous on each ray as $r$ tends to zero, is not an admissible expansion. We introduce the following definition.

Definition 6.1. A metric $C$ satisfying (3.3), (3.4) is said to be near-round at the vertex if there is a neighbourhood of $O$ where $r^{-1} \underline{\bar{c}_{i j}} \equiv \underline{\bar{d}_{i j}}$ and $r^{-2} \underline{c_{i j}} \delta^{i j} \equiv \bar{D}$ with $\bar{d}_{i j}$ and $\bar{D}$ being admissible series.

If $C$ is near-round at the vertex, $\bar{C}_{i j}$ has an analytic extension $\underline{C_{i j}}$ of the form, with $\underline{d_{i j}}$ being analytic extension of $\underline{\bar{d}_{i j}}$,

$$
\underline{C_{i j}} \equiv \delta_{i j}+\underline{c_{i j}}, \quad \underline{c_{i j}} \equiv y^{0} \underline{d_{i j}},
$$

therefore,

$$
\underline{\partial_{h} C_{i j}} \equiv 0 \quad \text { for } \quad y^{0}=0, \quad \text { hence } \quad \underline{\partial_{h} C^{i j}} \equiv 0 \quad \text { for } \quad y^{0}=0
$$

and

$$
\underline{C^{i j}} \equiv \delta^{i j}+y^{0} \underline{d^{i j}},
$$

with $\underline{d^{i j}}$ being some analytic functions. Hence

$$
\underline{C^{i j} c_{i j}} \equiv\left(y^{0}\right)^{2}\left\{D+\underline{d^{i j}} \underline{d_{i j}}\right\} .
$$

Using the definition of $\underline{X_{\alpha \beta}}$ we see that if $C$ is near-round at the vertex, then

$$
\begin{equation*}
\underline{X_{\alpha \beta}} \equiv y^{0} \underline{Y_{\alpha \beta}}, \quad \text { with } \quad \underline{Y_{i j}}:=\frac{1}{2}\left\{\underline{d_{i j}}+y^{0} \underline{\partial_{0} d_{i j}}+y^{h} \underline{\partial_{h} d_{i j}}\right\}, \quad \underline{Y_{i 0}} \equiv \underline{Y_{00}} \equiv 0 . \tag{6.17}
\end{equation*}
$$

Hence

$$
\operatorname{tr} X \equiv y^{0} \operatorname{tr} Y, \quad \text { with } \quad \operatorname{tr} Y \equiv \underline{C^{i j}} \underline{Y_{i j}} \equiv \frac{1}{2} \underline{C^{i j}}\left\{\underline{d_{i j}}+y^{0} \underline{\partial_{0} d_{i j}}+y^{h} \underline{\partial_{h} d_{i j}}\right\} .
$$

An elementary computation shows that $\operatorname{tr} Y$ is of the following form (with $Z$ being an analytic function)

$$
\operatorname{tr} Y \equiv y^{0} Z, \quad \text { hence } \quad \operatorname{tr} X \equiv\left(y^{0}\right)^{2} Z
$$

On the other hand,

$$
\underline{X_{j}^{i}} \equiv \underline{C^{i h}} \underline{X_{j h}}=y^{0} \underline{C^{i h}} \underline{Y_{j h}}:=y^{0} \underline{Y_{j}^{i}},
$$

therefore,

$$
|X|^{2} \equiv \underline{X_{j}^{i}} \underline{X_{i}^{j}} \equiv\left(y^{0}\right)^{2} \underline{Y_{j}^{i}} \underline{Y_{i}^{j}} .
$$

We deduce from these formulas that if $C$ is near-round at the vertex then $r^{-2} \overline{\operatorname{tr} X} \equiv \bar{Z}$ and $r^{-2}|\bar{X}|^{2} \equiv|\bar{Y}|^{2}$ are admissible series.

Theorem 6.2. A sufficient condition for $r^{-2}\left(\nu^{0}-1\right)$, with $\nu_{0}$ solution of the first wave-map gauge constraint, to have an admissible expansion is that the $C^{\omega}$ metric $C$ given by (3.3) which induces the basic characteristic data be near-round at the vertex.

Proof. Since $f:=\nu_{0}-1$ satisfies Eq. (6.8), $\phi:=r^{-2}\left(\nu_{0}-1\right)$ satisfies

$$
\begin{equation*}
r \partial_{1} \phi+(2+k) \phi+r^{-2} h=0 \tag{6.18}
\end{equation*}
$$

The expression (6.16) shows that for a metric $C$ round at the vertex $r^{-2} h$ admits an admissible development, the application of Corollary 5.1 gives the result.

## 7. The $C_{A}$ constraint

We have written in [1] the $C_{A}$ constraint in vacuum as

$$
\mathcal{C}_{A}=-\frac{1}{2}\left(\partial_{1} \xi_{A}+\tau \xi_{A}\right)+\widetilde{\nabla}_{B} \chi_{A}^{B}-\frac{1}{2} \partial_{A} \tau+\partial_{A}\left(\frac{1}{2} \bar{W}_{1}+\nu_{0} \partial_{1} \nu^{0}\right)
$$

where $\xi_{A}$ is defined as

$$
\begin{equation*}
\xi_{A}:=-2 \nu^{0} \partial_{1} \nu_{A}+4 \nu^{0} \nu_{C} \chi_{A}^{C}+\left(\bar{W}^{0}-\frac{2}{r} \nu^{0}\right) \nu_{A}+\bar{g}_{A B} \bar{g}^{C D}\left(S_{C D}^{B}-\widetilde{\Gamma}_{C D}^{B}\right) \tag{7.1}
\end{equation*}
$$

Using the first constraint we find

$$
\begin{equation*}
\nu_{0} \partial_{1} \nu^{0}+\frac{1}{2} \bar{W}_{1}=-a \tag{7.2}
\end{equation*}
$$

where $a$ is given by (6.14), hence

$$
\partial_{A}\left(a+\frac{1}{2} \tau\right) \equiv r^{-1} \partial_{A} F\left(\overline{\operatorname{tr} X},|\bar{X}|^{2}\right)
$$

where

$$
\begin{aligned}
F\left(\overline{\operatorname{tr} X},|\bar{X}|^{2}\right):=\frac{r \partial_{1} \overline{\operatorname{tr} X}+\overline{\operatorname{tr} X}+|\bar{X}|^{2}}{n-1+\overline{\operatorname{tr} X}} & +\frac{1}{2} \overline{\operatorname{tr} X} \equiv \\
\equiv & \frac{r \partial_{1} \overline{\operatorname{tr} X}+\frac{1}{2}\left\{(n+1) \overline{\operatorname{tr} X}+|\overline{\operatorname{tr} X}|^{2}\right\}+|\bar{X}|^{2}}{n-1+\overline{\operatorname{tr} X}}
\end{aligned}
$$

admits in a neighbourhood of $O$ an admissible development of minimal order 2.
We have:

$$
\begin{equation*}
\mathcal{C}_{A} \equiv-\frac{1}{2}\left(\partial_{1} \xi_{A}+\tau \xi_{A}\right)+\widetilde{\nabla}_{B} \chi_{A}^{B}-r^{-1} \partial_{A} F\left(\overline{\operatorname{tr} X},|\bar{X}|^{2}\right)=0 . \tag{7.3}
\end{equation*}
$$

7.1. Equations for $\xi_{A}$. We set $\xi_{1}=\xi_{0}=0$ on the cone and we define $\underline{\xi_{i}}$ by

$$
\begin{equation*}
\underline{\xi_{i}}:=\frac{\partial x^{\alpha}}{\partial y^{i}} \xi_{\alpha} \equiv \frac{\partial x^{A}}{\partial y^{i}} \xi_{A} \tag{7.4}
\end{equation*}
$$

It holds that

$$
\begin{equation*}
y^{i} \underline{\xi_{i}}=0 \tag{7.5}
\end{equation*}
$$

because (recall that $x^{1}=r, y^{i} \equiv r \Theta^{i}\left(x^{A}\right)$ )

$$
\begin{equation*}
y^{i} \frac{\partial x^{A}}{\partial y^{i}} \equiv r \frac{\partial y^{i}}{\partial x^{1}} \frac{\partial x^{A}}{\partial y^{i}} \equiv r \delta_{1}^{A}=0, \quad \frac{\partial}{\partial r} \frac{\partial y^{i}}{\partial x^{A}} \equiv \frac{1}{r} \frac{\partial y^{i}}{\partial x^{A}} \tag{7.6}
\end{equation*}
$$

We have

$$
\begin{equation*}
\frac{\partial y^{i}}{\partial x^{A}} \underline{\xi_{i}} \equiv \frac{\partial y^{i}}{\partial x^{A}} \frac{\partial x^{B}}{\partial y^{i}} \xi_{B} \equiv \delta_{A}^{B} \xi_{B} \equiv \xi_{A} \tag{7.7}
\end{equation*}
$$

The equation (7.7) implies that

$$
\partial_{1} \xi_{A} \equiv\left(\frac{\partial}{\partial r} \underline{\xi_{i}}+r^{-1} \underline{\xi_{i}}\right) \frac{\partial y^{i}}{\partial x^{A}}
$$

hence

$$
\mathcal{C}_{A} \equiv-\frac{1}{2} \frac{\partial y^{i}}{\partial x^{A}}\left\{\frac{\partial}{\partial r} \underline{\xi_{i}}+\underline{\xi_{i}}\left(r^{-1}+\tau\right)\right\}+\widetilde{\nabla}_{B} \chi_{A}^{B}-r^{-1} \partial_{A} F\left(\overline{\operatorname{tr} X},|\bar{X}|^{2}\right)=0
$$

Since $\operatorname{tr} X \equiv \underline{\operatorname{tr} X}$ is a scalar function and the equation of $C_{O}$ in the $x$ coordinates is $x^{0}=0$ and $y^{0}$ does not depend on $x^{A}$, it holds that

$$
\frac{\partial}{\partial x^{A}} \overline{\operatorname{tr} X} \equiv \overline{\frac{\partial}{\partial x^{A}} \operatorname{tr} X} \equiv \overline{\frac{\partial y^{i}}{\partial x^{A}} \frac{\partial}{\partial y^{i}} \frac{\operatorname{tr} X}{}} \equiv \frac{\partial y^{i}}{\partial x^{A}} \overline{\frac{\partial}{\partial y^{i}} \underline{\operatorname{tr} X}},
$$

analogously

$$
\frac{\partial}{\partial x^{A}} \overline{|X|^{2}} \equiv \frac{\partial y^{i}}{\partial x^{A}} \overline{\frac{\partial}{\partial y^{i}} \underline{\left.X\right|^{2}}}, \quad \frac{\partial}{\partial x^{A}} \overline{\overline{F\left(\operatorname{tr} X,|X|^{2}\right)}} \equiv \frac{\partial y^{i}}{\partial x^{A}} \overline{\frac{\partial}{\partial y^{i}} \underline{F\left(\operatorname{tr} X,|X|^{2}\right)} .}
$$

We now compute, with covariant derivatives $\widetilde{\nabla}$ taken in the Riemannian metric $\bar{g}_{A B} \equiv \bar{C}_{A B}$,

$$
\widetilde{\nabla}_{B} \chi_{A}^{B} \equiv \widetilde{\nabla}_{B}\left(\frac{1}{x^{1}} \bar{X}_{A}^{B}+\frac{1}{x^{1}} \delta_{A}^{B}\right) \equiv \frac{1}{x^{1}} \widetilde{\nabla}_{B} \bar{X}_{A}^{B}
$$

The Christoffel symbols $\widetilde{C}_{B C}^{A}$ of the Riemannian connection $\widetilde{\nabla}$ are equal (recall that $C^{B 0}=C^{B 1}=C^{00}=0$ ) to the trace on $C_{O}$ of the Christoffel symbols with the same indices of the spacetime metric $C$, hence denoting by ${ }^{(C)} \nabla$ the covariant derivative in the metric $C$

$$
\widetilde{\nabla}_{B} \chi_{A}^{B} \equiv \frac{1}{x^{1}} \overline{(C) \nabla_{B} X_{A}^{B}} .
$$

Since the $X_{A}^{B}$ are the only non-vanishing components of the tensor $X$, and due to the form chosen for the metric $C$ we find that

$$
\overline{{ }^{(C)} \nabla_{B} X_{A}^{B}}=\overline{{ }^{(C)} \nabla_{\alpha} X_{A}^{\alpha}}=\frac{\partial y^{i}}{\partial x^{A}} \overline{(C) \nabla_{\alpha} X_{i}^{\alpha}}
$$

and Eqs. (7.3) read

$$
\begin{equation*}
\left.\mathcal{C}_{A} \equiv \frac{\partial y^{j}}{\partial x^{A}} \frac{1}{r}\left\{-\frac{1}{2}\left[r \frac{\partial}{\partial r \underline{\xi_{j}}}+\underline{\xi_{j}}(n+\underline{\overline{\operatorname{tr} X}})\right]+\overline{\overline{(C) \nabla_{\alpha} X_{j}^{\alpha}}}-\overline{\frac{\partial}{\partial y^{j}} F\left(\operatorname{tr} X,|X|^{2}\right.}\right)\right\}=0 \tag{7.8}
\end{equation*}
$$

The parentheses constitute a linear diagonal operator on the $\underline{\xi}_{j}$ of the type considered in Lemma 5.3. Equating it to zero gives an equation with solution $\underline{\xi_{j}}$ an admissible series of minimal order 1 . We denote by $\underline{\Xi}_{j}$ the extension of $\underline{\xi_{i}}$ to spacetime, that is we have

$$
\begin{equation*}
\underline{\xi_{i}} \equiv \underline{\overline{\Xi_{i}}}, \tag{7.9}
\end{equation*}
$$

where $\underline{\Xi}_{i}$ are analytical functions beginning by linear terms.
7.2. Equations for $\underline{\nu_{i}}$. We now consider Eqs. (7.1), which read

$$
\begin{equation*}
\partial_{1} \nu_{A}+\left(\frac{1}{r}-\frac{1}{2} \bar{W}_{1}\right) \nu_{A}-2 \nu_{C} \chi_{A}^{C}-\frac{1}{2} \nu_{0} \bar{g}_{A B} \bar{g}^{C D}\left(S_{C D}^{B}-\widetilde{\Gamma}_{C D}^{B}\right)+\frac{1}{2} \nu_{0} \xi_{A}=0 . \tag{7.10}
\end{equation*}
$$

We set

$$
\begin{equation*}
\underline{\bar{g}_{0 i}} \equiv-\underline{\nu_{i}}+\lambda \underline{\bar{L}_{i}}, \quad \text { with } \quad \underline{\bar{L}_{i}} \equiv \underline{\bar{C}_{i j}} \underline{\bar{L}^{j}}, \quad \underline{\bar{L}^{j}} \equiv y^{j}, \tag{7.11}
\end{equation*}
$$

with $\underline{\nu_{i}}$ such that

$$
\begin{equation*}
\underline{\nu_{i}} \bar{L}^{i} \equiv \underline{\nu_{i}} y^{i}=0 \tag{7.12}
\end{equation*}
$$

that is, using (7.11),

$$
\lambda \equiv\left(\bar{L}_{i} \bar{L}^{i}\right)^{-1} \underline{\bar{g}_{0 j} \bar{L}^{j}} \equiv r^{-2} y^{j} \underline{\bar{g}_{0 j}} .
$$

Then (compare (3.5))

$$
\nu_{A} \equiv-\frac{\partial y^{i}}{\partial x^{A}} \underline{\bar{g}_{0 i}} \equiv \frac{\partial y^{i}}{\partial x^{A}} \underline{\nu_{i}} .
$$

Hence

$$
\partial_{1} \nu_{A} \equiv \frac{\partial y^{i}}{\partial x^{A}}\left(\partial_{1} \underline{\nu_{i}}+r^{-1} \underline{\nu_{i}}\right)
$$

We recall that

$$
\chi_{A}^{C} \equiv \frac{1}{r}\left(\bar{X}_{A}^{C}+\delta_{A}^{C}\right), \quad \text { i.e. } \quad r \nu_{C} \chi_{A}^{C} \equiv \nu_{C} \bar{X}_{A}^{C}+\nu_{A}
$$

Therefore, after product by $r$, the equations can be written as follows:

$$
\frac{\partial y^{i}}{\partial x^{A}}\left(r \partial_{1} \underline{\nu_{i}}-\frac{1}{2} r \bar{W}_{1} \underline{\nu_{i}}+\frac{1}{2} \nu_{0} r \underline{\xi_{i}}\right)-2 \bar{F}_{A}-\frac{1}{2} r \nu_{0} \bar{E}_{A}=0
$$

with

$$
\bar{F}_{A}:=\nu_{C} \bar{X}_{A}^{C}, \quad \bar{E}_{A}:=\bar{g}_{A B} \bar{g}^{C D}\left(S_{C D}^{B}-\widetilde{\Gamma}_{C D}^{B}\right)
$$

By definition it holds that

$$
\nu_{C} \bar{X}_{A}^{C} \equiv \overline{g_{0 C} X_{A}^{C}},
$$

and since $X_{A}^{C}$ are the only non-vanishing components of the mixed tensor $X$ in the coordinates $x$

$$
g_{0 C} X_{A}^{C}=g_{0 \lambda} X_{A}^{\lambda} \equiv \frac{\partial y^{\alpha}}{\partial x^{0}} \frac{\partial y^{\beta}}{\partial x^{A}} \underline{g_{\alpha \lambda}} \underline{X_{\beta}^{\lambda}}
$$

Recalling that $\underline{X_{i}^{0}} \equiv 0$ we find (using (4.11), $\overline{X_{i}^{j} L_{j}}=0$ )

$$
\bar{F}_{A} \equiv \nu_{C} \bar{X}_{A}^{C} \equiv-\frac{\partial y^{i}}{\partial x^{A}} \overline{\overline{g_{0 j} X_{i}^{j}}} \equiv \frac{\partial y^{i}}{\partial x^{A}} \overline{\nu_{j} X_{i}^{j}} .
$$

We then remark that $S_{C D}^{B}-\widetilde{C}_{C D}^{B}$ is the trace on $C_{O}$ of the difference of the components of the Christoffel symbols, $\eta_{\beta \gamma}^{\alpha}$ and $C_{\beta \gamma}^{\alpha}$, with these angular $x$ indices of the Minkowski metric $\eta$ and the metric $C$ :

$$
\bar{E}_{A}:=\bar{g}_{A B} \bar{g}^{C D}\left(S_{C D}^{B}-\widetilde{C}_{C D}^{B}\right), \quad \text { with } \quad E_{A} \equiv C_{A B} C^{C D}\left(\eta_{C D}^{B}-C_{C D}^{B}\right)
$$

Using the expressions of $\eta$ and $C$ and the vanishing of the Christoffel symbols of $\eta$ in $y$ coordinates we find

$$
\bar{E}_{A} \equiv \overline{C_{A \alpha} C^{\lambda \mu}\left(\eta_{\lambda \mu}^{\alpha}-C_{\lambda \mu}^{\alpha}\right)} \equiv \frac{\partial y^{i}}{\partial x^{A}} \overline{C_{i \alpha} C^{\lambda \mu}\left(\eta_{\lambda \mu}^{\alpha}-C_{\lambda \mu}^{\alpha}\right)} \equiv-\frac{\partial y^{i}}{\partial x^{A}} \overline{C_{i j} C^{\lambda \mu} C_{\lambda \mu}^{j}},
$$

with $C_{\lambda \mu}^{j}$ analytic functions, components of Christoffel symbols of the metric $C$ in $y$ coordinates, that is, using the values of the $y$ components of the metric $C$

$$
\underline{C_{i j} C^{\lambda \mu} C_{\lambda \mu}^{j}} \equiv \frac{1}{2} \underline{C^{h k}\left(\partial_{h} c_{i k}+\partial_{h} c_{i k}-\partial_{i} c_{h k}\right)}
$$

We recall from (6.15) that

$$
r \bar{W}_{1} \equiv-\nu_{0} \bar{g}^{A B} r^{2} s_{A B} \equiv-\nu_{0}\left\{n-1+\overline{c^{i j} \delta_{i j}}\right\}
$$

and we find that Eqs. (7.10) can be written as

$$
\frac{\partial y^{i}}{\partial x^{A}} \mathcal{L}_{i}=0
$$

where $\mathcal{L}_{i}$ is the following linear operator on $\underline{\nu_{i}}$

$$
\begin{equation*}
r \partial_{1} \underline{\nu_{i}}+\nu_{0}\left\{\frac{n-1}{2}+\frac{1}{2} \underline{\overline{c^{h k} \delta_{h k}}}\right\} \nu_{i}-2 \overline{\overline{\nu_{j} X_{i}^{j}}}+\frac{1}{2} \nu_{0} r \underline{\xi_{i}}-\frac{1}{2} r \nu_{0} \overline{\overline{C_{i j} C^{\lambda \mu} C_{\lambda \mu}^{j}}}=0 . \tag{7.13}
\end{equation*}
$$

We extend Lemma 5.3 as follows.

Lemma 7.1. If $k_{i}^{j}$ and $h_{i}$ are admissible series of minimal orders 1 , and the constant $k_{0} \geq 0$, then the $O D E$

$$
\begin{equation*}
r \partial_{1} \underline{\nu_{i}}+k_{0} \underline{\nu_{i}}+k_{i}^{j} \underline{\nu_{j}}=h_{i} \tag{7.14}
\end{equation*}
$$

admits a solution $\underline{\nu_{i}}$, which is also an admissible series of the same minimal order than $h$.

Recalling that $\nu_{0}-1$ is an admissible series of minimal order 2 we see that this lemma applies to (7.13). We have proved:

Theorem 7.1. If the basic characteristic data is induced by an analytic metric $C$ satisfying (3.3), (3.4), Eq. (7.13) admits one and only one solution $\underline{\nu_{i}}$ which is an admissible series of minimal order 2. We denote by $\underline{N_{i}}$ its spacetime extension.

We now prove:
Theorem 7.2. A sufficient condition for $r^{-2} \underline{\nu_{i}}$ to have an admissible expansion is that the $C^{\omega}$ metric $C$ given by (3.3) which induces the basic characteristic data be near-round at the vertex.

Proof. Using the relation between $X$ and $Y$ the linearity of $F$ in $\operatorname{tr} X$ and $|X|^{2}$ we see that for $C$ round at the vertex the equation satisfied by $\underline{\xi_{j}}$ reads

$$
\frac{1}{2}\left\{r \frac{\partial}{\partial r \underline{\xi_{j}}}+\underline{\xi_{j}}\left(n+r^{2} \bar{Z}\right)\right\}=h_{j}
$$

with

$$
h_{j}:=\overline{\overline{(C) \nabla_{\alpha}\left(\left(y^{0}\right)^{2} Y_{j}^{\alpha}\right)}}-\overline{\overline{\frac{\partial}{\partial y^{j}} F\left(\left(y^{0}\right)^{2} \operatorname{tr} Y,\left(y^{0}\right)^{4}|Y|^{2}\right)} .}
$$

We deduce from the linearity of $F$ in $\operatorname{tr} X$ and $|X|^{2}$ that $r^{-1} h_{j}$ admits an admissible expansion, the same holds therefore (see Corollary 5.1) for $r^{-1} \underline{\xi_{j}}$. The equation satisfied by $\underline{\nu_{j}}$ reads

$$
\begin{align*}
& r \partial_{1} \underline{\nu_{i}}+\nu_{0}\left\{\frac{n-1}{2}+\overline{\overline{r^{2} d^{h k} \delta_{h k}}}\right\} \underline{\nu_{i}}-2 \nu_{j} r^{2} \underline{Y_{i}^{j}}=h_{i},  \tag{7.15}\\
& h_{i}=-\frac{1}{2} \nu_{0} r \underline{\xi_{i}}+\frac{1}{4} r^{3} \nu_{0} \overline{C^{h k}\left(\partial_{h} d_{i k}+\partial_{h} d_{i k}-\partial_{i} d_{h k}\right)} .
\end{align*}
$$

An extension of Lemma 7.1 shows that $r^{-2} \underline{\nu_{i}}$ admits an admissible expansion because it is so of $h_{i}$.

## 8. The $\mathcal{C}_{0}$ constraint

The last unknown in $\bar{g}$, the only unknown in the constraint $\mathcal{C}_{0}$, is

$$
\bar{g}_{00} \equiv \underline{\bar{g}_{00}} .
$$

The constraint $\mathcal{C}_{0}$ has a simpler expression in terms of $\bar{g}^{11}$. Since $\bar{g}^{11}$ is linked to $\bar{g}_{00}$ by the identity

$$
\bar{g}^{01} \bar{g}_{00}+\bar{g}^{11} \bar{g}_{10}+\bar{g}^{A 1} \bar{g}_{A 0}=0
$$

we have

$$
\begin{equation*}
\bar{g}_{00} \equiv-\bar{g}^{11}\left(\nu_{0}\right)^{2}+\bar{g}^{A B} \nu_{B} \nu_{A} \equiv-\bar{g}^{11}\left(\nu_{0}\right)^{2}+\underline{\bar{C}^{i j}} \underline{\nu_{i}} \underline{\nu_{j}} . \tag{8.1}
\end{equation*}
$$

We have seen [1] that the $\mathcal{C}_{0}$ constraint can be written in vacuum as

$$
\begin{equation*}
\partial_{1} \zeta+(\kappa+\tau) \zeta+\frac{1}{2}\left\{\partial_{1} \bar{W}^{1}+(\kappa+\tau) \bar{W}^{1}+\widetilde{R}-\frac{1}{2} \bar{g}^{A B} \xi_{A} \xi_{B}+\bar{g}^{A B} \widetilde{\nabla}_{A} \xi_{B}\right\}=0 \tag{8.2}
\end{equation*}
$$

with

$$
\begin{gather*}
\zeta:=\left(\partial_{1}+\kappa+\frac{1}{2} \tau\right) \bar{g}^{11}+\frac{1}{2} \bar{W}^{1}  \tag{8.3}\\
\kappa \equiv \nu^{0} \partial_{1} \nu_{0}-\frac{1}{2}\left(\bar{W}_{1}+\tau\right), \quad \bar{W}_{1} \equiv \nu_{0} \bar{W}^{0}, \quad \bar{W}^{0} \equiv \bar{W}^{1} \equiv-r \bar{g}^{A B} s_{A B} \tag{8.4}
\end{gather*}
$$

8.1. Equation for $\zeta$. In the flat case it holds that

$$
\nu_{0, \eta}=1, \quad \tau_{\eta}=-\bar{W}_{1, \eta}=\frac{n-1}{r}, \quad \kappa_{\eta}=0 .
$$

The function $\zeta$ reduces to

$$
\begin{equation*}
\zeta_{\eta}:=\partial_{1} \bar{g}^{11}+\frac{1}{2} \tau_{\eta}\left(\bar{g}^{11}-1\right) \tag{8.5}
\end{equation*}
$$

and the equation for $\zeta$ reads, using $\xi_{A, \eta}=0$,

$$
\partial_{1} \zeta_{\eta}+\frac{n-1}{r} \zeta_{\eta}+\frac{1}{2}\left\{\frac{n-1}{r^{2}}-\frac{(n-1)^{2}}{r^{2}}+\widetilde{R}_{\eta}\right\}=0 .
$$

That is, using the scalar curvature of the $S^{n-1}$ round sphere of radius $r$ which is ${ }^{5}$

$$
\begin{equation*}
\widetilde{R}_{\eta}=r^{-2}(n-2)(n-1), \tag{8.6}
\end{equation*}
$$

the equation

$$
\partial_{1} \zeta_{\eta}+\frac{n-1}{r} \zeta_{\eta}=0
$$

has the only bounded solution $\zeta_{\eta} \equiv 0$. From (8.5) results we have then $\bar{g}_{\eta}^{11} \equiv 1$. We now study the general case.

We can write the equation to be satisfied by $\zeta$ as follows,

$$
\begin{gather*}
r \partial_{1}(r \zeta)+{ }^{(r \zeta)} k r \zeta+{ }^{(r \zeta)} h=0,  \tag{8.7}\\
{ }^{(r \zeta)} k:=r(\kappa+\tau)-1 \equiv r\left\{\nu^{0} \partial_{1} \nu_{0}+\frac{1}{2}\left(\tau-\overline{W_{1}}\right)\right\}-1,  \tag{8.8}\\
{ }^{(r \zeta)} h:=\frac{r^{2}}{2}\left\{\partial_{1} \bar{W}^{1}+(\kappa+\tau) \bar{W}^{1}+\widetilde{R}-\frac{1}{2} \bar{g}^{A B} \xi_{A} \xi_{B}+\bar{g}^{A B} \widetilde{\nabla}_{A} \xi_{B}\right\} .
\end{gather*}
$$

Hence

$$
\begin{equation*}
{ }^{(r \zeta)} h:=\frac{r^{2}}{2}\left\{\partial_{1} \bar{W}^{1}+\frac{1}{2}\left(\tau-\bar{W}_{1}\right) \bar{W}^{1}+\widetilde{R}+\nu^{0} \partial_{1} \nu_{0} \bar{W}^{1}-\frac{1}{2} \bar{g}^{A B} \xi_{A} \xi_{B}+\bar{g}^{A B} \widetilde{\nabla}_{A} \xi_{B}\right\} . \tag{8.9}
\end{equation*}
$$

We have shown that $\nu^{0}-1$ and $r \partial_{1} \nu_{0}$, hence also $r \nu^{0} \partial_{1} \nu_{0}$, admit admissible expansions of minimal order 2, and we have seen that $r \tau$ and $r \overline{W_{1}}$ are admissible series with terms of order zero $(n-1)$ and $-(n-1)$ respectively. Hence $k$ is an admissible series of the

[^4]zero-order term $n-2$, and we have
$$
{ }^{(r \zeta)} k=n-2+k_{1},
$$
where $k_{1}$ is an admissible series of minimal order 1.
We study the terms appearing in ${ }^{(r \zeta)} h$.
The term
\[

$$
\begin{equation*}
r^{2} \nu^{0} \partial_{1} \nu_{0} \bar{W}^{1} \equiv r \nu^{0} \partial_{1} \nu_{0} r \bar{W}^{1} \tag{8.10}
\end{equation*}
$$

\]

has an admissible expansion of minimal order 2 (see Lemma 5.4).
The term

$$
\begin{equation*}
r^{2} \bar{g}^{A B} \xi_{A} \xi_{B} \equiv r^{2} \underline{\bar{C}^{i j}} \underline{\xi_{i}} \underline{\xi_{j}} \tag{8.11}
\end{equation*}
$$

has an admissible expansion of order 4 because $\underline{\xi_{i}}$ has an admissible expansion of order 1 .
As for the term

$$
\bar{g}^{A B} \widetilde{\nabla}_{A} \xi_{B} \equiv \bar{C}^{A B} \widetilde{\nabla}_{A} \xi_{B}
$$

the Christoffel symbols $\widetilde{C}_{B C}^{A}$ of the Riemannian connection $\widetilde{\nabla}$ are equal (recall that $C^{B 0}=C^{B 1}=C^{00}=0$ ) to the trace on $C_{O}$ of the Christoffel symbols with the same indices of the spacetime metric $C$, hence denoting by ${ }^{(C)} \nabla$ the covariant derivative in the metric $C$

$$
\widetilde{\nabla}_{A} \xi_{B} \equiv \overline{(C) \nabla_{A} \Xi_{B}}
$$

Since the $\Xi_{B}$ are the only non-vanishing components of the vector $\Xi$, and due to the form chosen for the metric $C$ we find

$$
\begin{equation*}
\bar{C}^{A B} \overline{(C) \nabla_{A} \Xi_{B}}=\overline{\overline{C^{\alpha \beta}(C) \nabla_{\alpha} \Xi_{\beta}}} \tag{8.12}
\end{equation*}
$$

Hence the scalar $r^{2} \bar{C}^{A B} \widetilde{\nabla}_{A} \xi_{B}$ has an admissible expansion of minimal order 2.
We have seen that in the flat case

$$
\begin{equation*}
\bar{W}_{\eta}^{1} \equiv \eta^{\alpha \beta} \widehat{\Gamma}_{\alpha \beta}^{1} \equiv-\frac{n-1}{r} \tag{8.13}
\end{equation*}
$$

and

$$
\begin{equation*}
\partial_{1} \bar{W}_{\eta}^{1}+\tau \bar{W}_{\eta}^{1} \equiv\left\{\frac{n-1}{r^{2}}-\frac{(n-1)^{2}}{r^{2}}\right\}=-\widetilde{R}_{\eta} \tag{8.14}
\end{equation*}
$$

In the general case we compute

$$
\partial_{1} \bar{W}^{1}+\frac{1}{2}\left(\tau-\bar{W}_{1}\right) \bar{W}^{1} .
$$

Recall that

$$
\tau \equiv \frac{n-1}{r}+\frac{\overline{\operatorname{tr} X}}{r}
$$

set

$$
\bar{W}^{1} \equiv \bar{W}_{\eta}^{1}+F, \quad \text { with } \quad F:=\left(\bar{g}^{\alpha \beta}-\eta^{\alpha \beta}\right) \widehat{\Gamma}_{\alpha \beta}^{1}
$$

Using the values of the Christoffel symbols $\hat{\Gamma}_{\alpha \beta}^{1}$ and the components of $g$ and $\eta$ in $x$ coordinates we find

$$
\begin{aligned}
F \equiv- & \left(\bar{C}^{A B}-\eta^{A B}\right) x^{1} s_{A B} \equiv-\frac{1}{r}\left(\bar{C}^{\alpha \beta}-\eta^{\alpha \beta}\right) \eta_{\alpha \beta} \equiv \\
& \equiv \frac{1}{r}\left(n+1-\bar{C}^{\alpha \beta} \eta_{\alpha \beta}\right) \equiv \frac{1}{r}\left(n+1-\bar{C}^{\alpha \beta} \eta_{\alpha \beta}\right) \equiv \frac{1}{r}\left(n-\underline{\bar{C}^{i j} \eta_{i j}}\right) \equiv-\frac{1}{r} \underline{\bar{c}^{i j}} \delta_{i j}
\end{aligned}
$$

hence

$$
\bar{W}^{1} \equiv-\frac{n-1}{r}-\frac{\bar{c}^{i j} \delta_{i j}}{r}, \quad \partial_{1} \bar{W}^{1} \equiv \frac{n-1}{r^{2}}+\frac{\left(\bar{c}^{i j}-y^{h} \underline{\partial}_{h} \bar{c}^{i j}\right) \delta_{i j}}{r^{2}}
$$

and

$$
\frac{1}{2}\left(\tau-\bar{W}^{1}\right) \bar{W}^{1} \equiv-\left\{\frac{n-1}{r}+\frac{\overline{\operatorname{tr} X}+\bar{c}^{i j} \delta_{i j}}{2 r}\right\}\left\{\frac{n-1}{r}+\frac{1}{r} \bar{c}^{i j} \delta_{i j}\right\}
$$

Using the value of the scalar curvature $\widetilde{R}_{\eta}$ of the round sphere $S^{n-1}$ of radius $r$ we find that

$$
r^{2}\left\{\partial_{1} \bar{W}^{1}+\frac{1}{2}\left(\tau-\bar{W}^{1}\right) \bar{W}^{1}\right\} \equiv-r^{2} \widetilde{R}_{\eta}+\Phi
$$

where $\Phi$ is an admissible expansion of minimal order 2 ,

$$
\Phi:=-y^{h} \underline{\partial_{h} \bar{c}^{i j}} \delta_{i j}-(n-2) \underline{\bar{c}^{i j}} \delta_{i j}-\frac{1}{2}\left(n-1+\underline{\bar{c}^{h k} \delta_{h k}}\right)\left(\overline{\operatorname{tr} X}+\underline{\bar{c}^{i j} \delta_{i j}}\right) .
$$

To compute $r^{2} \widetilde{R}$, we use formulas given in [1]. The formulas (10.33) and (10.37) of [1] for a general metric in null adapted coordinates are

$$
\begin{gathered}
\bar{g}^{A B} \bar{R}_{A B} \equiv 2\left(\partial_{1}+\bar{\Gamma}_{11}^{1}+\tau\right)\left[\left(\partial_{1}+\bar{\Gamma}_{11}^{1}+\frac{\tau}{2}\right) \bar{g}^{11}+\bar{\Gamma}^{1}\right]+\widetilde{R}-2 \bar{g}^{A B} \bar{\Gamma}_{1 A}^{1} \bar{\Gamma}_{1 B}^{1}-2 \bar{g}^{A B} \widetilde{\nabla}_{A} \bar{\Gamma}_{1 B}^{1} \\
\bar{R}_{11} \equiv-\partial_{1} \tau+\bar{\Gamma}_{11}^{1} \tau-\chi_{A}^{B} \chi_{B}^{A}
\end{gathered}
$$

and

$$
\bar{S}_{01} \equiv-\frac{1}{2} \nu_{0} \bar{g}^{A B} \bar{R}_{A B}+\bar{R}_{1 A} \nu^{A}-\frac{1}{2} \nu_{0} \bar{g}^{11} \bar{R}_{11}
$$

In the case of the metric $C$, it holds that $C_{01}=1, C_{0 A}=0, C_{00}=-1$, and $C^{11}=1$. Hence

$$
{ }^{(C)} \Gamma_{11}^{1} \equiv{ }^{(C)} \Gamma_{1 A}^{1} \equiv 0, \quad{ }^{(C)} \Gamma^{1} \equiv-\frac{1}{2}\left(C^{A B} \partial_{1} C_{A B}+C^{A B} \partial_{0} C_{A B}\right),
$$

and the above formulas reduce to (recall that $\widetilde{R} \equiv{ }^{(C)} \widetilde{R}$ ):

$$
\begin{aligned}
& \bar{C}^{A B}(C) \bar{R}_{A B} \equiv-\left(\partial_{1}+\tau\right)\left[\tau+\bar{C}^{A B} \overline{\partial_{0} C_{A B}}\right]+\widetilde{R} \\
& \bar{R}_{11} \equiv-\partial_{1} \tau-\chi_{A}^{B} \chi_{B}^{A}
\end{aligned}
$$

and

$$
-2^{(C)} \bar{S}_{01} \equiv \bar{g}^{A B} \bar{R}_{A B}+\bar{R}_{11}
$$

from which we deduce

$$
\widetilde{R} \equiv 2 \partial_{1} \tau+\tau^{2}+\left(\partial_{1}+\tau\right) \bar{C}^{A B} \overline{\partial_{0} C_{A B}}+\chi_{A}^{B} \chi_{B}^{A}-2^{(C)} \bar{S}_{01}
$$

We have

$$
{ }^{(C)} \bar{S}_{01} \equiv-\left({ }^{(C)} \underline{\bar{S}_{00}}+r^{-1} y^{i}{ }^{(C)} \underline{\bar{S}_{0 i}}\right)
$$

Since $C$ is an analytic metric in a neighbourhood of $O,{ }^{(C)} \underline{\bar{S}_{\alpha \beta}}$ admit admissible expansions and, hence $r^{2}{ }^{(C)} \bar{S}_{01}$ also admits an admissible expansion of minimal order 2.

Recall that

$$
\tau \equiv \frac{n-1}{r}+\frac{\overline{\operatorname{tr} X}}{r} \quad \text { and } \quad|\chi|^{2}=\frac{|\bar{X}|^{2}+2 \overline{\operatorname{tr} X}+n-1}{r^{2}}
$$

hence

$$
r^{2}\left\{2 \partial_{1} \tau+\tau^{2}+\chi_{A}^{B} \chi_{B}^{A}\right\} \equiv(n-1)(n-2)+2(n-1) \overline{\operatorname{tr} X}+2 r \partial_{1} \overline{\operatorname{tr} X}+(\overline{\operatorname{tr} X})^{2}+|\bar{X}|^{2}
$$

Finally we remark that $\partial_{0} C_{A B}$ are the non-vanishing components of the Lie derivative of the metric $C$ with respect to the vector $m$ with $x$ components $m^{0}=1$, $m^{1}=m^{A}=0$, hence with $y$ components $\underline{m^{0}}=-1, \underline{m^{i}}=0$ that

$$
C^{A B} \partial_{0} C_{A B} \equiv C^{\alpha \beta} \mathcal{L}_{m} C_{\alpha \beta} \equiv \underline{C^{\alpha \beta} \partial_{0} C_{\alpha \beta}}
$$

hence

$$
\bar{C}^{A B} \overline{\partial_{0} C_{A B}} \equiv \overline{C^{i j}} \overline{\partial_{0} C_{i j}} \equiv \overline{C^{i j}} \overline{\partial_{0} c_{i j}}
$$

has an admissible expansion of minimal order 1 and $r^{2}\left(\partial_{1}+\tau\right) \bar{C}^{A B} \overline{\partial_{0} C_{A B}}$ has an admissible expansion of minimal order 2.

We have proved that

$$
r^{2} \widetilde{R} \equiv r^{2} \widetilde{R}_{\eta}+\Psi
$$

where

$$
\Psi \equiv 2(n-1) \overline{\operatorname{tr} X}+2 r \partial_{1} \overline{\operatorname{tr} X}+(\overline{\operatorname{tr} X})^{2}+|\bar{X}|^{2}+r^{2}\left(\partial_{1}+\tau\right) \underline{\overline{C^{i j}}} \underline{\overline{\partial_{0} c_{i j}}}
$$

has an admissible expansion of minimal order 2. Hence

$$
r^{2}\left\{\partial_{1} \bar{W}^{1}+\frac{1}{2}\left(\tau-\bar{W}^{1}\right) \bar{W}^{1}+\widetilde{R}\right\} \equiv \Phi+\Psi
$$

In conclusion, we have shown that ${ }^{(r \zeta)} h$ has an admissible expansion of minimal order 2 , the same is therefore true for $r \zeta$.
8.2. Equation for $\alpha:=\bar{g}^{11}-1$. The definition of $\zeta$ gives for $\alpha:=\bar{g}^{11}-1$ an equation which reads

$$
\begin{equation*}
r \partial_{1} \alpha+r\left(\kappa+\frac{1}{2} \tau\right) \alpha+\frac{r}{2}\left(2 \kappa+\tau+\bar{W}_{1}-2 \zeta\right)=0 . \tag{8.15}
\end{equation*}
$$

Theorem 8.1. If the basic characteristic data is induced in a neighbourhood of $O$ by an analytic metric $C$ satisfying (3.3), (3.4), then Eq. (8.15) admits in this neighbourhood one and only one solution $\alpha$ which is an admissible series of minimal order 2.

This implies that $\bar{g}_{00}+1$ is also an admissible series of minimal order 2.
Proof. Using the definition of $\kappa$ in (8.4), the equation (8.15) reads

$$
r \partial_{1} \alpha+r\left(\kappa+\frac{1}{2} \tau\right) \alpha+r\left(\nu^{0} \partial_{1} \nu_{0}-\zeta\right)=0
$$

that is,

$$
\begin{equation*}
r \partial_{1} \alpha+{ }^{(\alpha)} k \alpha+{ }^{(\alpha)} h=0, \tag{8.16}
\end{equation*}
$$

with

$$
\begin{equation*}
{ }^{(\alpha)} k:=r\left(\nu^{0} \partial_{1} \nu_{0}-\frac{1}{2} \bar{W}_{1}\right), \quad{ }^{(\alpha)} h:=r\left(\nu^{0} \partial_{1} \nu_{0}-\zeta\right) . \tag{8.17}
\end{equation*}
$$

Previous results show that this equation is of a form to which Lemma 5.3 applies with ${ }^{(\alpha)} k_{0}=\frac{n-1}{2}$ and ${ }^{(\alpha)} h$ of minimal order 2. It has therefore a solution $\alpha$, admissible series of minimal order 2 .

The identity (8.1) shows the property of $\bar{g}_{00}+1$.

Theorem 8.2. If in addition to the hypothesis of the previous theorem the given metric $C$ is near-round at the vertex, then $r^{-2}\left(\bar{g}_{00}+1\right)$ is an admissible series in a neighbourhood of the vertex.

Proof. The result will follow from the proof that ${ }^{(\alpha)} h$ given in (8.17) is such that $r^{-2(\alpha)} h$ is an admissible series. By previous results, it remains only to prove that $r^{-2}(r \zeta)$ has an admissible expansion, hence that it is so for $r^{-2(r \zeta)} h$. We have

$$
\begin{equation*}
r^{-2(r \zeta)} h:=\frac{1}{2}\left\{r^{-2}(\Phi+\Psi)+\nu^{0} \partial_{1} \nu_{0} \bar{W}^{1}-\frac{1}{2} \bar{g}^{A B} \xi_{A} \xi_{B}+\bar{g}^{A B} \widetilde{\nabla}_{A} \xi_{B}\right\} \tag{8.18}
\end{equation*}
$$

We check that if $C$ is near-round at the vertex, the various terms studied above are such that the required condition is satisfied. Indeed,

$$
\nu^{0} \partial_{1} \nu_{0} \bar{W}^{1} \equiv \nu^{0} r^{-1} \partial_{1} \nu_{0} r \bar{W}^{1}
$$

has an admissible expansion because it is so for $r^{-1} \partial_{1} \nu_{0}$ and $r \bar{W}^{1}$.

$$
\bar{g}^{A B} \xi_{A} \xi_{B} \equiv \underline{\bar{C}^{i j}} \underline{\xi_{i}} \underline{\xi_{j}}, \quad \bar{g}^{A B} \widetilde{\nabla}_{A} \xi_{B} \equiv \underline{\overline{C^{\alpha \beta}(C)} \nabla_{\alpha} \Xi_{\beta}}
$$

have admissible expansions because $\underline{\xi_{i}}$ does.
The assumptions on $c^{i j}$ and the identity $y^{h} \partial_{h} r \equiv r$ show that

$$
y^{h} \partial_{h} \underline{\bar{c}^{i j} \delta_{i j}} \equiv r^{2}\left(y^{h} \partial_{h} \underline{\bar{d}^{i j} \delta_{i j}}+2 \underline{\bar{d}^{i j} \delta_{i j}}\right),
$$

hence $r^{-2} \Phi$, and $r^{-2} \Psi$, with $\Phi$ and $\Psi$ given above have admissible expansions.
It remains to show the property for

$$
{ }^{(C)} \bar{S}_{01} \equiv-\left({ }^{(C)} \underline{\bar{S}_{00}}+r^{-1} y^{i(C)} \underline{\bar{S}_{0 i}}\right) .
$$

Since $C$ is an analytic metric in a neighbourhood of $O,{ }^{(C)} \underline{\bar{S}}_{00}$ has an admissible expansion. Denoting by ${ }^{(C)} \frac{K_{i j}}{0} \equiv-\frac{1}{2} \underline{\partial_{0} C_{i j}}$ the second fundamental form of $C$ relative to the slicing of $R^{n+1}$ by $y^{0}=$ constant, we know that ${ }^{6}$,

$$
{ }^{(C)} \underline{\bar{S}_{0 i}} \equiv \underline{\partial_{i} \operatorname{tr}^{(C)} K}-{ }^{(C)} \underline{\bar{\nabla}_{j}\left(C^{j h(C)} K_{i h}\right)} .
$$

We have

$$
{ }^{(C)} \underline{\bar{\nabla}_{j}\left(C^{j h(C)} K_{i h}\right)} \equiv \underline{C^{j h}} \underline{\left(\partial_{j}{ }^{(C)} K_{i h}-C_{j i}^{k}{ }^{(C)} K_{i k}-C_{j h}^{k(C)} K_{j k}\right) . ~}
$$

The functions ${ }^{(C)} \underline{K_{i j}}$ are analytic and the Christoffel symbols $\underline{C_{j i}^{k}}$ are products by $y^{0}$ of analytic functions, while elementary computations give
hence using $y^{i} \underline{c_{i h}}=0$ and $\delta^{j h} \underline{c_{j h}}=\left(y^{0}\right)^{2} Z$. We deduce from these results that $r^{-1} y^{i(C)} \underline{\bar{S}_{0 i}}$ also has an admissible expansion.

[^5]
## 9. Conclusions

9.1. An existence theorem. We have shown that when the metric $C$ given by (3.3) which induces on $C_{O}$ the basic characteristic data $\widetilde{g}$ (i.e. $\bar{g}_{A B} \equiv \bar{C}_{A B} \equiv \widetilde{C}_{A B}$ ) is analytic, then the functions $\nu_{0}, \underline{\nu_{i}}, \underline{\bar{g}_{00}}$ have admissible expansions. We have shown that if moreover $C$ is near-round at the vertex (definition 6.2), then the functions $r^{-2}\left(\nu_{0}-1\right), r^{-1} \underline{\nu_{i}}$, and $r^{-2}\left(\bar{g}_{00}+1\right)$ have also admissible expansions. These results imply the following theorem (the notations are those of Section 1):

Theorem 9.1. If the metric $C$ given by (3.3), (3.4) which induces the basic characteristic data on the cone $C_{O}^{T}$ is smooth everywhere, and moreover analytic and nearround in a neighbourhood of the vertex, then there exists a number $T_{0}>0$ such that the wave-gauge reduced vacuum Einstein equations with characteristic initial determined by $C$ and the solution of the wave-map gauge constraints have a solution in $Y_{O}^{T_{0}}$ which induces on $C_{O}^{T_{0}}$ the same quadratic form as $C$.

Proof. It results from the formulae

$$
\begin{gather*}
\left.\underline{\bar{g}_{00}} \equiv \bar{g}_{00}, \quad \underline{\bar{g}_{0 i}} \equiv-\left\{\bar{g}_{00}+\nu_{0}\right)\right\} r^{-1} y^{i}-\underline{\nu_{i}}, \quad \text { with } \quad y^{i} \underline{\nu_{i}} \equiv 0,  \tag{9.1}\\
\underline{\bar{g}_{i j}}-\delta_{i j} \equiv\left\{\bar{g}_{00}+1+2\left(\nu_{0}-1\right)\right\} r^{-2} y^{i} y^{j}+r^{-1}\left(y^{i} \underline{\nu_{j}}+y^{j} \underline{\nu_{i}}\right)+\underline{c_{i j}}, \tag{9.2}
\end{gather*}
$$

and the theorems of previous sections that $\underline{\bar{g}_{00}}+1, \bar{g}_{0 i}$, and $\bar{g}_{i j}-\delta_{i j}$ have admissible expansions, hence are the trace on $C_{O}$ of analytic functions. We apply the CagnacDossa theorem.

This theorem and the results of [1] lead then to the following sufficient conditions for the existence of a solution of the full Einstein equations.

Theorem 9.2. If the metric $C$ given by (3.3) which induces the basic characteristic data on the cone $C_{O}$ is smooth, and analytic and near-round at the vertex, there exists a vacuum Einsteinian spacetime $\left(Y_{O}^{T_{0}}, g\right)$ which induces on $C_{O}^{T_{0}}$ the same quadratic form as $C$. The solution is locally geometrically unique.

JMM was supported by the French ANR grant BLAN07-1_201699 entitled "LISA Science", and also in part by the Spanish MICINN project FIS2009-11893. PTC was supported in part by the Polish Ministry of Science and Higher Education grant Nr N N201 372736.

## Резюме

И. Шоке-Брюа, П.Т. Хрусъцель, Х.М. Мартин-Гарсиа. Теорема существования решения задачи Коши о световом конусе для вакуумных уравнений Эйнштейна с «почтизакругленными» аналитическими данными.

Представлен класс характеристических релятивистски обобщенных исходных данных, удовлетворяющих условию почтизакругленности на вершине светового конуса. Показано, что для любых подобных аналитических данных существует соответствующее решение для вакуумных уравнений Эйнштейна, определенное в окрестности вершины в световом конусе будущего.

Ключевые слова: характеристическая задача Коши о световом конусе, вакуумные уравнения Эйнштейна.

## References

1. Choquet-Bruhat Y., Chrusciel P.T., Martin-Garcia J.M. The Cauchy problem on a characteristic cone for the Einstein equations in arbitrary dimensions // Ann. Henri Poincare. - 2011. - V. 12, No 3. - P. 419-482.
2. Rendall A.D. Reduction of the characteristic initial value problem to the Cauchy problem and its applications to the Einstein equations // Proc. R. Soc. London A. - 1990. V. 427. - P. 221-239.
3. Damour T., Schmidt B. Reliability of perturbation theory in general relativity // J. Math. Phys. - 1990. - V. 31. - P. 2441-2453.
4. Choquet-Bruhat Y., Chruściel P.T., Martín-García J.M. An existence theorem for the Cauchy problem on a characteristic cone for the Einstein equations // Contemporary Mathematics. - 2011. - V. 554. - P. 73-81.
5. Chruściel P.T., Jezierski J. On free general relativistic initial data on the light cone. arXiv:1010.2098 [gr-qc]. - 2010. - URL: http://arxiv.org/pdf/1010.2098v1.pdf.
6. Choquet-Bruhat Y., DeWitt-Morette C. Analysis, manifolds and physics, Part II. - Amsterdam: North-Holland Publ.g Co., 2000. - 560 p.
7. Choquet-Bruhat Y. General relativity and the Einstein equations. - Oxford: Oxford Univ. Press, 2009. - 840 p.

## Поступила в редакцию 28.11.10

[^6]
[^0]:    ${ }^{1}$ For previous writing of these equations in the case of two intersecting surfaces in four-dimensional spacetime, see Rendall [2] and Damour - Schmidt [3].

[^1]:    ${ }^{2}$ Smooth means $C^{m}$, with $m$ being some integer depending on the problem at hand and the considered function. In particular $C^{\infty}$ and $C^{\omega}$ (real analytic functions) are smooth.

[^2]:    ${ }^{3}$ Recall that in arbitrary coordinates $x^{I}$ the Lie derivative reads

    $$
    \left(\mathcal{L}_{\ell} \widetilde{C}\right)_{H K} \equiv \ell^{I} \partial_{I} \widetilde{C}_{H K}+\widetilde{C}_{H I} \partial_{K} \ell^{I}+\widetilde{C}_{K I} \partial_{H} \ell^{I}
    $$

[^3]:    ${ }^{4}$ Recall that we underline components in the $y$ coordinates and overline restrictions to $C_{O}$.

[^4]:    ${ }^{5}$ See for instance [6, p. 140].

[^5]:    ${ }^{6}$ See for instance [7, Chapter 6].

[^6]:    Choquet-Bruhat, Yvonne - PhD, Full Member of the French Academy of Sciences, Paris, France.

    Шоке-Брюа, Ивонн - доктор наук, полноправный член Французской академии наук, г. Париж, Франция.

    E-mail: ycb@ihes.fr
    Chruściel, Piotr T. - PhD, Professor, University of Vienna, Vienna, Austria.
    Хрусьцель, Петр Т. - доктор наук, профессор Венского университета, г. Вена, Австрия.

    E-mail: piotr.chrusciel@univie.ac.at
    Martín-García, José Maria - PhD, Postdoctoral Research Fellow, Paris Institute of Astrophysics, Paris, France; Universe and Theories Laboratory, Meudon, France.

    Мартин-Гарсиа, Хосе Мария - доктор наук, научный сотрудник Парижского астрофизического института, г. Париж, Франция; научный сотрудник Лаборатории Вселенной и теорий о ней, Мёдон (пригород Парижа), Франция.

