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AN EXISTENCE THEOREM FOR THE CAUCHY PROBLEM ON THE LIGHT-CONE FOR THE VACUUM EINSTEIN EQUATIONS WITH NEAR-ROUND ANALYTIC DATA

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Abstract

A class of characteristic general relativistic initial data satisfying a *near-roundness* condition at the tip of a light-cone is introduced. It is shown that for any such analytic data there exists a corresponding solution of the vacuum Einstein equations defined in a future neighborhood of the vertex.

Key words: characteristic Cauchy problem on a light-cone, vacuum Einstein equations.

Introduction

In the paper [1] we studied the Cauchy problem for the Einstein equations with data on a characteristic cone C_O . We used the tensorial splitting of the Ricci tensor of a Lorentzian metric g on a manifold V as the sum of a quasidiagonal hyperbolic system acting on g and a linear first order operator acting on a vector H, called the wavegauge vector. The vector H vanishes if g is in wave gauge; that is, if the identity map is a wave map from (V,g) onto (V,\widehat{g}) , with \widehat{g} being some given metric, which we have chosen to be Minkowski. The data needed for the reduced PDEs is the trace, which we denote by \overline{q} , of q on C_Q . However, because of the constraints, the intrinsic, geometric, data is a degenerate quadratic form \tilde{g} on C_O . Given \tilde{g} , the trace \overline{g} is determined through a hierarchical system of ordinary differential equations¹ along the rays of C_O , deduced from the contraction of the Einstein tensor with a tangent to the rays, which we have written explicitly and solved. We have called these equations the wave map gauge constraints and shown that they are necessary and sufficient conditions for the solutions of the hyperbolic system to satisfy the full Einstein equations. We have also proved local geometric uniqueness of a solution g of the vacuum Einstein equations inducing a given \tilde{g} (for details see [1]). Further references to previous works on the problem at hand can be found in [1].

Existence theorems known for quasilinear wave equations with data on a characteristic cone give also existence theorems for the Einstein equations, if the initial data is Minkowski in a neighbourhood of the vertex. For more general data problems arise due to the apparent discrepancy between the functional requirements on the characteristic data of the hyperbolic system and the properties of the solutions of the constraints, due to the singularity of the cone C_O at its vertex O. The aim of this work is to make progress towards resolving this issue, and provide a sufficient condition for the validity of an existence theorem in a neighbourhood of O under conditions alternative to the fast-decay conditions of [4]. More precisely, we prove that analytic initial data arising

¹For previous writing of these equations in the case of two intersecting surfaces in four-dimensional spacetime, see Rendall [2] and Damour-Schmidt [3].

from a metric satisfying (3.3), (3.4) together with the "near-roundness" condition of Definition 6.1 lead to a solution of the vacuum Einstein equations to the future of the light-cone.

1. Cauchy problem on a characteristic cone for quasilinear wave equations

The reduced Einstein equations in wave-map gauge and Minkowski target are a quasi-diagonal, quasi-linear second order system for a set v of scalar functions v^{I} , I = 1, ..., N, on \mathbf{R}^{n+1} of the form

$$A^{\lambda\mu}(y,v)D^{2}_{\lambda\mu}v + f(y,v,Dv) = 0, \quad y = (y^{\lambda}) \in \mathbf{R}^{n+1}, \quad n \ge 2, \quad f = (f^{I}).$$
(1.1)

If the target is the Minkowski metric and takes in the coordinates y^{α} the canonical form

$$\eta \equiv -(dy^0)^2 + \sum_{i=1}^n (dy^i)^2, \qquad (1.2)$$

then,

$$Dv = \frac{\partial v^I}{\partial y^{\lambda}}, \quad D^2_{\lambda\mu}v = \frac{\partial^2 v^I}{\partial y^{\lambda}\partial y^{\mu}}, \quad \lambda, \, \mu = 0, 1, \dots, n.$$
 (1.3)

We will underline components in these y^{α} coordinates.

In the case of the Einstein equations the functions $A^{\lambda\mu} \equiv g^{\lambda\mu}$ do not depend directly on y, they are analytic in v in an open set $W \subset \mathbf{R}^N$. For $v \in W$ the quadratic form $g^{\lambda\mu}$ is of Lorentzian signature. The functions f^I are analytic in $v \in W$ and $Dv \in \mathbf{R}^{(n+1)N}$, they do not depend directly on y in vacuum.

The characteristic cone C_O of vertex O for a Lorentzian metric g is the set covered by future directed null geodesics issued from O. We choose coordinates y^{α} such that the coordinates of O are $y^{\alpha} = 0$ and the components $A^{\lambda\mu}(0,0)$ take the diagonal Minkowskian values, $(-1, 1, \ldots, 1)$. If v is $C^{1,1}$ in a neighbourhood U of O and takes its values in W there is an eventually smaller neighbourhood of O, still denoted U, such that $C_O \cap U$ is an n dimensional manifold, differentiable except at O, and there exist in U coordinates $y := (y^{\alpha}) \equiv (y^0, y^i, i = 1, \ldots, n)$ in which C_O is represented by the equation of a Minkowskian cone with vertex O,

$$C_O := \{r - y^0 = 0\}, \quad r := \left\{\sum (y^i)^2\right\}^{1/2}, \tag{1.4}$$

and the null rays of C_O represented by the generators of the Minkowskian cone, i.e. tangent to the vector ℓ with components $\underline{\ell^0} = 1$, $\underline{\ell^i} = r^{-1}y^i$. Inspired by this result and following previous authors we will set the Cauchy problem for the equations (1.1) on a characteristic cone as the search of a solution which takes given values on a manifold represented by an equation of the form (1.4), that is, a set v such that

$$\overline{v} = \varphi, \tag{1.5}$$

where overlining means restriction to C_O . The function φ takes its values in W and is such that ℓ is a null vector for \overline{A} , i.e. when $\overline{A} \equiv \overline{g}$

$$\ell^{\mu}\ell^{\nu}\overline{g}_{\mu\nu} = \overline{g}_{00} + 2r^{-1}y^{i}\overline{g}_{0i} + r^{-2}y^{i}y^{j}\overline{g}_{ij} = 0.$$
(1.6)

We use the following notations:

$$C_O^T := C_O \cap \{ 0 \le t := y^0 \le T \}, Y_O := \{ y^0 > r \}, \text{ the interior of } C_O, Y_O^T := Y_O \cap \{ 0 \le y^0 \le T \}.$$

and we set

$$\Sigma_{\tau} := C_O \cap \{y^0 = \tau\}, \quad \text{diffeomorphic to } S^{n-1}, \\ S_{\tau} := Y_O \cap \{y^0 = \tau\}, \quad \text{diffeomorphic to the ball } B^{n-1}$$

We recall the following theorem, which applies in particular to the reduced Einstein equations.

Theorem 1.1. Consider the problem (1.1), (1.5). Suppose that:

1. There is an open set $U \times W \subset \mathbf{R}^{n+1} \times \mathbf{R}^N$, $Y_O^T \subset U$ where the functions $g^{\lambda \mu}$ are smooth in y and v. The function f is smooth² in $y \in U$ and $v \in W$ and in $Dv \in \mathbf{R}^{(n+1)N}$.

2. For $(y, v) \in U \times W$ the quadratic form $g^{\lambda \mu}$ has Lorentzian signature; it takes the Minkowskian values for y = 0 and v = 0. It holds that $\varphi(O) = 0$.

3a. The function φ takes its values in W. The cone C_O^T is null for the metric $g^{\lambda\mu}(y,\varphi)$.

3b. φ is the trace on C_O^T of a smooth function in U.

Then there is a number $0 < T_0 \leq T < +\infty$ such that the problem (1.1), (1.5) has one and only one solution v in $Y_O^{T_0}$ which can be extended by continuity to a smooth function defined on a neighbourhood of the origin in \mathbf{R}^{n+1} .

If φ is small enough in appropriate norms, then $T_0 = T$.

2. Null adapted coordinates

It has been shown (see [1] and references therein) that the constraints are easier to solve in coordinates x^{α} adapted to the null structure of C_O , defined by

$$x^{0} = r - y^{0}, \quad x^{1} = r \quad \text{and} \ x^{A} = \mu^{A}(r^{-1}y^{i}),$$
 (2.1)

where A = 2, ..., n, local coordinates on the sphere S^{n-1} , or angular polar coordinates. Conversely

$$y^{0} = x^{1} - x^{0}, \quad y^{i} = r\Theta^{i}(x^{A}) \quad \text{with} \quad \sum_{i=1}^{n} \Theta^{i}(x^{A})^{2} = 1$$

In the x coordinates the Minkowski metric (1.2) reads

$$\eta \equiv -(dx^0)^2 + 2dx^0 dx^1 + (x^1)^2 s_{n-1}, \qquad (2.2)$$

with

 S_r

$$s_{n-1} := s_{AB} dx^A dx^B$$
, the metric of the round sphere S^{n-1} .

Recall that in these coordinates the non zero Christoffel symbols of the Minkowski metric are, with S_{BC}^{A} , the Christoffel symbols of the metric s,

$$\hat{\Gamma}^{B}_{1A} \equiv \frac{1}{x^{1}} \delta^{B}_{A} , \quad \hat{\Gamma}^{B}_{AC} \equiv S^{B}_{AC} , \quad \hat{\Gamma}^{0}_{AB} \equiv -x^{1} s_{AB} , \quad \hat{\Gamma}^{1}_{AB} \equiv -x^{1} s_{AB} .$$
 (2.3)

In the general case, the null geodesics issued from O have still equation $x^0 = 0$, $x^A = \text{constant}$, so that $\ell := \frac{\partial}{\partial x^1}$ is tangent to those geodesics. The trace \overline{g} on C_O of the spacetime metric g that we are going to construct is such that $\overline{g}_{11} = 0$ and $\overline{g}_{1A} = 0$; we use the notation

$$\overline{g} \equiv \overline{g}_{00} (dx^0)^2 + 2\nu_0 dx^0 dx^1 + 2\nu_A dx^0 dx^A + \overline{g}_{AB} dx^A dx^B, \qquad (2.4)$$

²Smooth means C^m , with m being some integer depending on the problem at hand and the considered function. In particular C^{∞} and C^{ω} (real analytic functions) are smooth.

We emphasize that our assumption that \overline{g} is given by (2.4) is no geometric restriction for a metric g to have such a trace on a null cone $x^0 = 0$.

The Lorentzian metric g induces on C_O a degenerate quadratic form \widetilde{g} which reads in coordinates x^1, x^A

$$\tilde{g} \equiv \tilde{g}_{AB} dx^A dx^B, \tag{2.5}$$

i.e. $\tilde{g}_{11} \equiv \tilde{g}_{1A} \equiv 0$ while $\tilde{g}_{AB} dx^A dx^B \equiv \overline{g}_{AB} dx^A dx^B$ is an x^1 -dependent Riemannian metric on S^{n-1} induced on each Σ_t by \tilde{g} , we denote it by \tilde{g}_{Σ} . While \tilde{g} is intrinsically defined, it is not so for \overline{g}_{00} , ν_0 , ν_A , they are gauge-dependent quantities.

Note that \tilde{g} has a more complicated expression in coordinates y^i on C_O . Since the inclusion mapping of C_O in the coordinates y^{α} is $y^0 = r$, hence $\frac{\partial y^0}{\partial y^i} = \frac{y^i}{r}$, it holds that

$$\widetilde{g} \equiv \underline{\widetilde{g}_{ij}} dy^i dy^j, \quad \text{with} \quad \underline{\widetilde{g}_{ij}} \equiv r^{-2} y^i y^j \underline{\overline{g}_{00}} + r^{-1} (y^j \underline{\overline{g}_{0i}} + y^i \underline{\overline{g}_{0j}}) + \underline{\overline{g}_{ij}}.$$
(2.6)

For Theorem 1.1, to apply to the wave-gauge reduced Einstein equations, the components of the initial data in the y coordinates must be the trace on C_O of smooth spacetime functions. The solution of the reduced equations satisfy the full Einstein equations if and only if these initial data satisfy the wave-map gauge constraints. We have constructed in [1] these data as solutions of ODE in adapted null x coordinates, which are admissible coordinates for \mathbf{R}^{n+1} only for r > 0. The change of coordinates from x to y, smooth for r > 0, is recalled below; the components of a spacetime tensor Tin the coordinates x are denoted by $T_{\alpha\beta}$, while in the coordinates y they are denoted by $T_{\alpha\beta}$.

Lemma 2.1. It holds that:

$$T_{00} \equiv \underline{T}_{00}, \quad T_{11} \equiv \underline{T}_{00} + 2\frac{y^{i}}{r} \underline{T}_{0i} + \frac{y^{i}}{r} \frac{y^{j}}{r} \underline{T}_{ij}, \quad T_{01} \equiv -(\underline{T}_{00} + \underline{T}_{0i}\Theta^{i}),$$

$$T_{0A} \equiv -r \frac{\partial \Theta^{i}}{\partial x^{A}} \underline{T_{0i}}, \quad T_{1A} \equiv r \frac{\partial \Theta^{i}}{\partial x^{A}} (\underline{T_{0i}} + \Theta^{j} \underline{T_{ij}}), \quad T_{AB} \equiv \underline{T_{ij}} r^{2} \frac{\partial \Theta^{i}}{\partial x^{A}} \frac{\partial \Theta^{j}}{\partial x^{B}}.$$

Conversely, if $T_{1A} \equiv T_{11} \equiv 0$, then

$$\underline{T_{00}} \equiv T_{00}, \quad \underline{T_{0i}} \equiv -(T_{00} + T_{01})r^{-1}y^i - T_{0A}\frac{\partial x^A}{\partial y^i},$$
$$\underline{T_{ij}} = (T_{00} + 2T_{01})r^{-2}y^iy^j + T_{0A}r^{-1}(y^i\frac{\partial x^A}{\partial y^j} + y^j\frac{\partial x^A}{\partial y^i}) + T_{AB}\frac{\partial x^A}{\partial y^i}\frac{\partial x^B}{\partial y^j}.$$

Hereinafter, we shall often abbreviate partial derivatives as follows

$$\partial_0 \equiv \frac{\partial}{\partial x^0}, \quad \partial_1 \equiv \frac{\partial}{\partial x^1}, \quad \partial_A \equiv \frac{\partial}{\partial x^A},$$

 $\underline{\partial}_0 \equiv \frac{\partial}{\partial y^0}, \quad \underline{\partial}_i \equiv \frac{\partial}{\partial y^i}.$

3. Characteristic data

3.1. Basic (intrinsic) characteristic data. The basic data on a characteristic cone for the Einstein equation is a degenerate quadratic form. We will define this data as the degenerate quadratic form \tilde{C} induced by a given Lorentzian spacetime metric C which admits this cone as a null cone. By C_O , as before, we denote the manifold $x^0 \equiv r - y^0 = 0$. If we take the y^i as coordinates on C_O , it holds that, see (2.6),

$$\widetilde{C} \equiv \underline{\widetilde{C}_{ij}} dy^i dy^j \quad \text{with} \quad \underline{\widetilde{C}_{ij}} \equiv \underline{\overline{C}_{ij}} + r^{-1} (y^j \underline{\overline{C}_{i0}} + y^i \underline{\overline{C}_{j0}}) + r^{-2} y^i y^j \underline{\overline{C}_{00}} , \qquad (3.1)$$

where $\overline{C}_{\alpha\beta}$ are the components in the y coordinates of the trace \overline{C} of C on C_O (not to be mistaken with the induced quadratic form \widetilde{C}). Assuming that C_O is a null cone for C with generators $\overline{\underline{\ell}}^0 = 1$, $\overline{\underline{\ell}}^i = \frac{y^i}{r}$, (1.6) implies that the quadratic form \widetilde{C} is degenerate,

$$\frac{y^i}{r}\frac{y^j}{r}\frac{\widetilde{C}_{ij}}{\widetilde{C}_{ij}} \equiv \overline{\underline{C}_{00}} + 2\frac{y^i}{r}\frac{\overline{\underline{C}}_{0i}}{\widetilde{\underline{C}}_{0i}} + \frac{y^i}{r}\frac{y^j}{r}\frac{\overline{\underline{C}}_{ij}}{\widetilde{\underline{C}}_{ij}} = 0.$$
(3.2)

Lemma 3.1. Two spacetime metrics C and C' with components linked by

$$\underline{\overline{C}_{ij}}' := \underline{\overline{C}_{ij}} + r^{-1}(a_i y_j + a_j y_i) + r^{-2} \alpha y^i y^j, \quad \underline{\overline{C}_{i0}}' := \underline{\overline{C}_{i0}} - a_i, \quad \underline{\overline{C}_{00}}' := \underline{\overline{C}_{00}} - \alpha,$$

with arbitrary a_i and α induce on C_O the same quadratic form \widetilde{C} , i.e. $\underline{\widetilde{C}_{ij}}' \equiv \underline{\widetilde{C}_{ij}}$.

Proof. Elementary calculation using the identity written above.

In what follows, to simplify computations we will make the restrictive condition that

$$\underline{C_{0i}} = 0, \quad \underline{C_{00}} = -1, \quad \text{i.e.} \quad C := -(dy^0)^2 + \underline{C_{ij}} dy^i dy^j.$$
 (3.3)

The set $\{y^0 = r\}$ is then a null cone for the metric *C*, with generator $\underline{\ell^0} = 1$, $\underline{\ell^i} = \frac{y^i}{r}$, if and only if

$$y^i \overline{\underline{C}_{ij}} = y^j \tag{3.4}$$

(compare [5]).

The general relation between components in coordinates y and adapted null coordinates x^1 , x^A gives

$$\widetilde{C} \equiv \widetilde{C}_{AB} dx^A dx^B$$
 with $\widetilde{C}_{AB} \equiv \overline{C}_{AB}$, $\widetilde{C}_{1A} = \widetilde{C}_{11} = 0$,

with \widetilde{C}_{AB} being the components of a $x^1 \equiv r$ -dependent Riemannian metric on the sphere S^{n-1}

$$-\left(\underline{C_{00}} + \frac{y^i}{r}\underline{C_{0i}}\right) \equiv C_{01} = 1, \quad C_{0A} \equiv -\frac{\partial y^i}{\partial x^A}\underline{C_{0i}} = 0, \quad \underline{C_{00}} \equiv C_{00} = -1.$$

This metric C is also such that

$$C^{00} \equiv C^{0A} \equiv C^{1A} \equiv 0, \quad C^{01} \equiv C^{11} \equiv 1,$$

while \overline{C}^{AB} are the elements of the inverse of the positive definite quadratic form with components \overline{C}_{AB} .

3.2. Full characteristic data. We have seen in [1] that the trace \overline{g} of a Lorentzian metric g satisfying the reduced Einstein equations is a solution of the full Einstein equations if and only if it satisfies the wave map gauge constraints. These constraints $C_{\alpha} = 0$ are deduced in vacuum from the identity satisfied by the Einstein tensor S:

$$\overline{\ell}^{\,\,
ho}\,\overline{S}_{lphaeta}\equiv \mathcal{C}_{lpha}+\mathcal{L}_{lpha}$$

where \mathcal{L}_{α} is linear and homogeneous in the wave gauge vector H while \mathcal{C}_{α} depends only on \overline{g} and its derivatives among C_O and the given target \hat{g} . Given \tilde{g} , i.e. $\overline{g}_{AB} \equiv \overline{C}_{AB}$, $\overline{g}_{1A} = \overline{g}_{11} = 0$, the remaining components $\nu_0 \equiv \overline{g}_{01}$, $\nu_A \equiv \overline{g}_{0A}$, \overline{g}_{00} are determined by the constraints and limit conditions at the vertex O which can always be satisfied

by choice of coordinates (see [1]). The Cagnac – Dossa theorem applies to components in the y coordinates. Lemma 2.1 gives

$$\overline{g}_{00} \equiv \overline{g}_{00}, \quad \overline{g}_{0i} \equiv -(\overline{g}_{00} + \nu_0)r^{-1}y^i - \underline{\nu_i}, \quad \text{with} \quad \underline{\nu_i} := \nu_A \frac{\partial x^A}{\partial y^i}, \quad (3.5)$$

$$\underline{\overline{g}_{ij}} = (\overline{g}_{00} + 2\nu_0)r^{-2}y^i y^j + r^{-1}(y^i \underline{\nu_j} + y^j \underline{\nu_i}) + \overline{g}_{AB} \frac{\partial x^A}{\partial y^i} \frac{\partial x^B}{\partial y^j},$$

while, for the chosen metric C and $\overline{g}_{AB} \equiv \overline{C}_{AB}$

$$\overline{\underline{C}}_{ij} \equiv r^{-2}y^i y^j + \overline{g}_{AB} \frac{\partial x^A}{\partial y^i} \frac{\partial x^B}{\partial y^j}.$$

Therefore,

$$\underline{\overline{g}}_{ij} = \underline{\overline{C}}_{ij} + (\overline{g}_{00} + 2\nu_0 - 1)r^{-2}y^i y^j + r^{-1}(y^i \underline{\nu}_j + y^j \underline{\nu}_j).$$
(3.6)

4. Null second fundamental form

We have defined in I the null second fundamental form of (C_O, \tilde{g}) as the tensor χ on C_O defined by the Lie derivative³ with respect to the vector ℓ of the degenerate quadratic form \tilde{g} , namely in the coordinates x^1, x^A :

$$\chi_{AB} := \frac{1}{2} (\mathcal{L}_{\ell} \widetilde{g})_{AB} \equiv \frac{1}{2} \partial_1 \overline{g}_{AB}, \qquad (4.1)$$

$$\chi_{A1} := \frac{1}{2} (\mathcal{L}_{\ell} \widetilde{g})_{A1} = 0, \quad \chi_{11} := \frac{1}{2} (\mathcal{L}_{\ell} \widetilde{g})_{11} = 0.$$
(4.2)

In view of the application of the Cagnac-Dossa theorem, we look for smooth extensions. We define a smooth spacetime vector field L, vanishing at O and with trace colinear with $\ell = \frac{\partial}{\partial x^1}$ on C_O , by its components in the x^{α} and y^{α} coordinates respectively:

$$L := y^{\lambda} \frac{\partial}{\partial y^{\lambda}} \equiv x^{0} \frac{\partial}{\partial x^{0}} + x^{1} \frac{\partial}{\partial x^{1}}, \quad \text{hence} \quad \overline{L} \equiv x^{1} \ell \equiv r \ell.$$

We assume that the metric C is smooth in U, a neighbourhood of O in \mathbb{R}^{n+1} , i.e. its components $\underline{C_{ij}}$ are of class C^m , with m as large as necessary in the considered context, functions of the y^{α} . We define a symmetric C^{m-1} 2-tensor X, identically zero in the case where $C \equiv \eta$, the Minkowski metric, by:

$$X := \frac{1}{2}\mathcal{L}_L C - C. \tag{4.3}$$

In y coordinates one has⁴, using $y^i C_{ij} = y^j$

$$\underline{X_{00}} \equiv \underline{X_{0i}} \equiv 0, \quad \underline{X_{ij}} \equiv \frac{1}{2} \{ y^0 \underline{\partial_0 C_{ij}} + y^h \underline{\partial_h C_{ij}} \}.$$
(4.4)

with

$$y^i \underline{\partial_0 C_{ij}} = 0,$$

³Recall that in arbitrary coordinates x^I the Lie derivative reads $(\mathcal{L}_{\ell} \widetilde{C})_{HK} \equiv \ell^I \partial_I \widetilde{C}_{HK} + \widetilde{C}_{HI} \partial_K \ell^I + \widetilde{C}_{KI} \partial_H \ell^I.$

⁴Recall that we underline components in the y coordinates and overline restrictions to C_O .

and, using $\partial_h y^i = \delta_h^i$

$$y^{i}y^{h}\underline{\partial_{h}C_{ij}} = y^{h}\underline{\partial_{h}}(y^{i}\underline{C_{ij}}) - y^{h}\underline{C_{ij}}\underline{\partial_{h}}y^{i} = 0,$$

$$(4.5)$$

which imply

$$\underline{L^i X_{ij}} \equiv y^i \underline{X_{ij}} = 0. \tag{4.6}$$

In x coordinates we find, using the values of the components $C_{0\alpha}$ and $C_{1\alpha}$ of the metric C, that the tensor X obeys the key properties

$$X_{\mu 0} = 0, \quad X_{\mu 1} = 0, \tag{4.7}$$

while

$$X_{AB} \equiv \frac{1}{2} (x^0 \partial_0 C_{AB} + x^1 \partial_1 C_{AB}) - C_{AB}.$$
 (4.8)

Hence X_{AB} reduces on the null cone C_O to

$$\overline{X}_{AB} \equiv \frac{1}{2}x^1 \partial_1 \overline{C}_{AB} - \overline{C}_{AB} \equiv x^1 \chi_{AB} - \overline{g}_{AB}.$$
(4.9)

By X we still denote the mixed C^{m-1} tensor on spacetime obtained from X by lifting an index with the contravariant associate of C; its y components are the C^{m-1} functions

$$2\underline{X_{\alpha}^{\gamma}} \equiv \underline{C^{\gamma\beta}} \{ y^{0} \underline{\partial_{0} C_{\alpha\beta}} + y^{i} \underline{\partial_{i} C_{\alpha\beta}} \},\$$

hence C being given by (3.3),

$$\underline{X_i^j} \equiv \frac{1}{2} \underline{C^{jh}} \{ y^0 \underline{\partial_0 C_{ih}} + y^k \underline{\partial_k C_{ih}} \}, \quad \underline{X_i^0} \equiv \underline{X_0^j} \equiv \underline{X_0^0} \equiv 0, \tag{4.10}$$

 and

$$\underline{X_i^j L_j} \equiv 0, \tag{4.11}$$

where the index of L has been lowered with the metric C, so that this is equivalent to (4.6). In x coordinates X_A^C are the only non-vanishing components of X. Their traces on C_O are

$$\overline{X}_{A}^{C} \equiv \frac{1}{2} x^{1} \overline{g}^{BC} \partial_{1} \overline{g}_{AB} - \delta_{A}^{C} \equiv x^{1} \chi_{A}^{C} - \delta_{A}^{C},$$

hence

$$\chi_A^C := \frac{1}{2} \overline{g}^{BC} \partial_1 \overline{g_{AB}} = \frac{1}{x^1} (\overline{X}_A^C + \delta_A^C), \qquad (4.12)$$

and

$$\tau := \frac{1}{2}\overline{g}^{AB}\partial_1\overline{g}_{AB} = \frac{\overline{\operatorname{tr} X}}{x^1} + \frac{n-1}{x^1}.$$
(4.13)

The trace of the tensor X is the C^{m-1} function

$$\operatorname{tr} X \equiv \underline{X_{\alpha}^{\alpha}} \equiv X_{\lambda}^{\lambda} \equiv C^{AB} X_{AB}.$$
(4.14)

On the light cone C_O it holds that

$$\overline{\operatorname{tr} X} \equiv \overline{\underline{X}}_{i}^{i} \equiv \overline{g}^{AB} \overline{X}_{AB} \equiv \frac{x^{1}}{2} \overline{g}^{AB} \partial_{1} \overline{g}_{AB} - (n-1), \qquad (4.15)$$

$$|\chi|^2 := \chi^C_A \chi^A_C \equiv \frac{1}{(x^1)^2} \left\{ \overline{X^\alpha_\beta} \, \overline{X^\beta_\alpha} + 2\overline{\operatorname{tr} X} + n - 1 \right\}.$$
(4.16)

5. A criterium: admissible series

To show that the integration of the constraints, which appear as ODE in x^1 , leads to traces on the cone of smooth spacetime functions, we shall use the following lemma, introduced by Cagnac (unpublished) for formal series, but used here for real analytic functions, a special class C^{ω} of C^{∞} functions.

Lemma 5.1. A function is the trace \overline{f} on C_O^T of a spacetime function f analytic in $U \cap Y_O^T$, U is a neighbourhood of O, if and only if it admits on $U \cap C_O^T$ a convergent expansion of the form

$$\overline{f} \equiv f_0 + \sum_{p=1}^{\infty} \overline{f}_p r^p \tag{5.1}$$

with

$$\overline{f}_p \equiv \overline{f}_{p,i_1\dots i_p} \Theta^{i_1} \dots \Theta^{i_p} + \overline{f}'_{p,i_1\dots i_{p-1}} \Theta^{i_1} \dots \Theta^{i_{p-1}},$$
(5.2)

where f_0 , $f_{p,i_1...i_p}$ and $f'_{p,i_1...i_{p-1}}$ are numbers. Such a series is called an admissible series. A coefficient f_p of the form (5.2) is called an admissible coefficient of order p.

Proof. If f is analytic it admits an expansion in Taylor series

$$f \equiv \sum_{p=0}^{\infty} f_{\alpha_1 \dots \alpha_p} y^{\alpha_1} \dots y^{\alpha_p}, \quad f_{\alpha_1 \dots \alpha_p} := \frac{1}{p!} \frac{\partial^p f}{\partial y^{\alpha_1} \dots \partial y^{\alpha_p}}(O).$$
(5.3)

One goes from the formulas (5.3) to (5.1), (5.2) by replacing y^i by $r\Theta^i$ and y^0 by r, and conversely, in $\Omega \cap C_O$ or in Ω .

Remark 5.1. The identity (5.1) is equivalent to saying that f is of the form $\overline{f} = f_1 + rf_2$, with f_1 and f_2 analytic functions of y^i .

We say that an admissible series is of minimal order q if the coefficients f_p are identically zero for p < q.

Proposition 5.1. If the metric C is analytic and satisfies the conditions (3.3), (3.4) then the functions $\overline{\operatorname{tr} X}$ and $|X|^2$ are admissible series of minimal orders 2 and 4 respectively.

The following lemmas will be very useful when integrating the constraints.

Lemma 5.2. If f_p and h_q are admissible coefficients of order p and q respectively, then $f_p + h_p$ and f_ph_q are admissible coefficients of order p and p + q respectively.

Proof. Elementary computation of $(f_p + h_p)r^p$ and $f_p h_q r^{p+q}$ replacing $r\Theta^i$ by y^i and r^2 by $\Sigma_i(y^i)^2$.

Suppose that f and h are admissible series of minimal orders q_f and q_h . The following are easy-to-check consequences of the lemma:

1) fh is an admissible series of minimal order $q_f + q_h$;

2) if $q_f = q_h$, then f + h is an admissible series of the same minimal order;

3) if $f(0) \neq 0$ and $q_f = 0$, then 1/f is an admissible series of also minimal order 0; 4) $r\partial_1 f$ is an admissible series of minimal order q_f , unless $q_f = 0$ and then it has a larger minimal order. **Lemma 5.3.** If k and h are admissible series with h of minimal order $q_h \ge 1$ and the constant $k_0 \equiv k(0) \ge 0$, then the ODE

$$r\partial_1 f + kf = h \tag{5.4}$$

admits one and only one solution f which is also an admissible series of the same minimal order q_h as h. The result extends to $q_h = 0$ if $k_0 > 0$.

Proof. Expand

$$f = \sum_{p=0}^{\infty} f_p r^p, \quad k = \sum_{p=0}^{\infty} k_p r^p, \quad h = \sum_{p=q_h}^{\infty} h_p r^p, \quad (5.5)$$

hence

$$r\partial_1 f = \sum_{p=1}^{\infty} p f_p r^p,$$

plug into the ODE (5.4) and proceed to identifications.

We obtain by equating to zero the constant term

$$k_0 f_0 = h_0, (5.6)$$

a relation which can be satisfied when $h_0 \neq 0$ only when $k_0 \neq 0$. We first consider the case where $h_0 = 0$, i.e. $q_h \geq 1$, and take $f_0 = 0$. We get the successive equalities

$$f_1 + k_0 f_1 = h_1$$
, i.e. $f_1 = \frac{h_1}{1 + k_0}$, (5.7)

and the recurrence relation, using $f_0 = 0$,

$$(p+k_0)f_p + \sum_{q=1}^{p-1} k_q f_{p-q} = h_p .$$
(5.8)

For $p < q_h$, we have $h_p = 0$ and the recurrence relation gives $f_p = 0$. Therefore, the leading admissible coefficients of f and h are always related by

$$f_{q_h} = \frac{h_{q_h}}{q_h + k_0}.$$
 (5.9)

We assume the series for k and h converge for all directions Θ^i and radius cr < 1; that is, we assume that there exists a constant c such that

$$|k_p| < c^p, \text{ and } |h_p| < c^p.$$
 (5.10)

Since $k_0 \ge 0$ we have

$$|f_1| \le |\frac{h_1}{1+k_0}| < \frac{c}{1+k_0} \le c.$$

Assume now that

$$|f_p| < c^p \quad \text{for} \quad p < p_0, \tag{5.11}$$

then from the iteration we get, for larger values of p, the inequality

$$|f_p| < \frac{p}{p+k_0}c^p \le c^p$$
 . (5.12)

The bounds on $|f_p|$ show that the series for f also converges. It is an admissible series of minimal order $q_f = q_h$.

When $q_h = 0$, i.e. $h_0 \neq 0$ and $k_0 \neq 0$ we take

$$f_0 = \frac{h_0}{k_0}$$

and we set

$$F = f - f_0.$$

It satisfies the equation

$$r\partial_1 F + kF = H$$
, with $H := h - kf_0$. (5.13)

We have

and we apply the previous result to F.

Corollary 5.1. If f and h are admissible series related by (5.4) and $p + k_0 > 0$, and $r^{-p}h$ is an admissible series then $r^{-p}f$ is an admissible series of the same minimal order.

 $H_0 = 0$

Proof. Set $f = r^p \phi$. If f satisfies (5.4) then ϕ satisfies the equation

$$r\partial_1\phi + (p+k)\phi = r^{-p}h.$$

Remark 5.2. The following example is a case of a differential equation of the form (5.4) with $q_h = 1$, but k_0 is a negative integer, which does not admit as a solution an admissible series. Let

$$k = \frac{1}{r-1} = -1 - r - r^2 - r^3 - \cdots, \quad h = \frac{r}{1-r^2} = r + r^3 + r^5 + \cdots,$$

We can solve the ODE explicitly,

$$f = \frac{r}{r-1} \left(f_{\infty} + \log \frac{r+1}{r} \right)$$

with f_{∞} an arbitrary integration constant, which cannot be expanded in powers of r near 0. However if we change k to r/(r-1), then k_0 changes from -1 to 0, the problem disappears. Remark that the problem also disappears if we change h to $r^2/(1-r^2)$, i.e. $q_h = 2$.

In what follows, we will assume the metric C, of the form (3.3) and satisfying (3.4) is analytic, takes Minkowskian values at the vertex O, and is such that the components of its trace on C_O satisfy

$$\underline{\overline{C}}_{ij} \equiv \delta_{ij} + \underline{\overline{c}}_{ij}, \quad \underline{\overline{C}}^{ij} \equiv \delta^{ij} + \underline{\overline{c}}^{ij}, \quad (5.14)$$

where \overline{c}_{ij} and \overline{c}^{ij} have admissible expansions of minimal order 2 while $\overline{\partial_0 c_{ih}}$ has an admissible expansion of minimal order 1. The definition (4.10) implies then that

$$\overline{\underline{X}_{i}^{j}} \equiv \frac{1}{2} \underline{C^{jh}} \left\{ r \,\overline{\underline{\partial_{0} \, c_{ih}}} + y^{k} \,\overline{\underline{\partial_{k} \, c_{ih}}} \right\}$$
(5.15)

has an admissible expansion of minimal order 2.

6. The first wave-map gauge constraint

We have deduced our first constraint(see [1]) from the identity

$$\overline{\ell}^{\beta} \overline{S}_{1\beta} \equiv \overline{R}_{11} \equiv -\partial_1 \tau + \nu^0 \partial_1 \nu_0 \tau - \frac{1}{2} \tau (\overline{\Gamma}_1 + \tau) - \chi^B_A \chi^A_B, \tag{6.1}$$

with

$$\overline{\Gamma}_1 \equiv \overline{W}_1 + \overline{H}_1, \quad \overline{W}_1 \equiv -\nu_0 \,\overline{g}^{AB} r s_{AB}. \tag{6.2}$$

Hence for the first wave-map gauge constraint in vacuum we have the equation

$$\mathcal{C}_1 := -\partial_1 \tau + \nu^0 \partial_1 \nu_0 \tau - \frac{1}{2} \tau (\tau - \nu_0 \overline{g}^{AB} r s_{AB}) - \chi^B_A \chi^A_B = 0.$$
(6.3)

When \overline{g}_{AB} is known this equation reads as a first order differential equation for ν_0

$$\nu^{0}\partial_{1}\nu_{0} = \tau^{-1}\partial_{1}\tau + \frac{1}{2}(\tau - \nu_{0}\overline{g}^{AB}rs_{AB}) + \tau^{-1}\chi^{B}_{A}\chi^{A}_{B}.$$
(6.4)

It can be written as a linear equation for $\nu^0 - 1$,

$$\partial_1 \nu^0 + a(\nu^0 - 1) + b = 0, \tag{6.5}$$

with

$$a := \tau^{-1} \partial_1 \tau + \frac{1}{2} \tau + \tau^{-1} |\chi|^2, \quad |\chi|^2 \equiv \chi^B_A \chi^A_B, \tag{6.6}$$

$$b := a - \frac{1}{2} \overline{g}^{AB} r s_{AB}. \tag{6.7}$$

In the flat case $\overline{g}^{AB} = \eta^{AB}$, $\tau = \frac{n-1}{r}$, $\chi^B_A = \frac{1}{r} \delta^B_A$ the equation reduces to:

$$\partial_1 \nu^0 + \frac{1}{2} \left(\nu^0 - 1\right) \frac{n-1}{r} = 0;$$

it has one solution tending to 1 when r tends to zero, $\nu_0 = 1$. In the general case (6.5) reads, with $f := \nu^0 - 1$,

$$r\partial_1 f + kf + h = 0, \quad k := ar, \quad h := br = ar - \frac{1}{2} \overline{g}^{AB} r^2 s_{AB}.$$
 (6.8)

Recall that $x^1 \equiv r$ and

au

$$\chi_A^C \equiv \frac{1}{2} \,\overline{g}^{BC} \partial_1 \overline{g_{AB}} = \frac{1}{r} \left(\overline{X}_A^C + \delta_A^C \right). \tag{6.9}$$

Hence

$$|\chi|^{2} = \frac{|\overline{X}|^{2} + 2\overline{\operatorname{tr} X} + n - 1}{r^{2}},$$
$$= \frac{\overline{\operatorname{tr} X}}{r} + \frac{n - 1}{r}, \quad \tau^{-1} = \frac{r}{n - 1 + \overline{\operatorname{tr} X}},$$
(6.10)

where $\overline{\operatorname{tr} X}$ is an admissible series of minimal order 2. The function $\left\{1 + \frac{1}{n-1}\overline{\operatorname{tr} X}\right\}^{-1}$ is the trace of a C^{ω} function as long as $1 + \frac{1}{n-1}\operatorname{tr} X$ does not vanish, hence always in a neighbourhood of O since $\operatorname{tr} X$ vanishes there.

It holds that

$$\partial_1 \tau \equiv -\frac{n-1+\overline{\operatorname{tr} X}}{r^2} + \frac{\partial_1 \overline{\operatorname{tr} X}}{r}, \qquad (6.11)$$

$$\tau^{-1}\partial_1\tau \equiv -\frac{1}{r} + \frac{\partial_1 \overline{\operatorname{tr} X}}{n-1 + \overline{\operatorname{tr} X}}.$$
(6.12)

Also we can write

$$\tau^{-1}|\chi|^2 \equiv \frac{|\overline{X}|^2 + 2\overline{\operatorname{tr} X} + n - 1}{r(n - 1 + \overline{\operatorname{tr} X})} \equiv \frac{|\overline{X}|^2 + \overline{\operatorname{tr} X}}{r(n - 1 + \overline{\operatorname{tr} X})} + \frac{1}{r}.$$
(6.13)

Finally computation gives

$$k \equiv ar \equiv \frac{n-1}{2} + \frac{r\partial_1 \overline{\operatorname{tr} X} + \overline{\operatorname{tr} X} + |\overline{X}|^2}{n-1 + \overline{\operatorname{tr} X}}.$$
(6.14)

We see that in a neighbourhood of O, $k - \frac{n-1}{2}$ admits an admissible development of minimal order 2.

On the other hand, since in the x coordinates $\eta_{1\alpha} = 0$, $r^2 s_{AB} = \eta_{AB}$ and we have assumed that $\overline{C}^{00} = \overline{C}^{0A} = 0$, $\overline{C}^{01} = 1$, $\overline{g}^{AB} = \overline{C}^{AB}$, we have

$$\overline{g}^{AB}r^2s_{AB} \equiv \overline{g}^{AB}\eta_{AB} \equiv \overline{C}^{AB}\eta_{AB} \equiv \overline{C}^{\alpha\beta}\eta_{\alpha\beta} - 2.$$

Hence using now the values of $C_{\alpha\beta}$

$$\frac{1}{2}\overline{g}^{AB}r^{2}s_{AB} \equiv \frac{1}{2}\overline{(1+C^{ij}\delta_{ij}-2)} \equiv \frac{n-1}{2} + \frac{1}{2}\overline{c^{ij}\delta_{ij}},$$
(6.15)

where $\overline{c^{ij}\delta_{ij}}$ has an admissible development of minimal order 2. We conclude that

$$h \equiv \frac{r\partial_1 \overline{\operatorname{tr} X} + \overline{\operatorname{tr} X} + |\overline{X}|^2}{(n-1+\overline{\operatorname{tr} X})} - \frac{1}{2} \frac{\overline{c^{ij}\delta_{ij}}}{(n-1+\overline{\operatorname{tr} X})}$$
(6.16)

admits also such a development. Lemma 5.3 applies, and we have proved the following theorem:

Theorem 6.1. If the basic characteristic data are induced on C_O by a C^{ω} (i.e. analytic) metric of the form (3.3), hence satisfying (2.4), then $\nu^0 - 1$ admits an admissible expansion of minimal order 2, hence is the trace in a neighbourhood of O of a C^{ω} spacetime function. Then $\nu^0 = \overline{N}^0$ with $N^0 \in C^{\omega}$ and $N^0(O) = 1$. In a neighbourhood of O, it holds that $\nu_0 = \overline{N}_0$, $N_0 = (N^0)^{-1}$.

In the expression of the characteristic initial data in y coordinates there appears $r^{-2}(\nu_0 - 1)$, which, though being continuous on each ray as r tends to zero, is not an admissible expansion. We introduce the following definition.

Definition 6.1. A metric *C* satisfying (3.3), (3.4) is said to be near-round at the vertex if there is a neighbourhood of *O* where $r^{-1}\overline{c_{ij}} \equiv \overline{d_{ij}}$ and $r^{-2}\underline{c_{ij}}\delta^{ij} \equiv \overline{D}$ with $\overline{d_{ij}}$ and \overline{D} being admissible series.

If C is near-round at the vertex, \overline{C}_{ij} has an analytic extension \underline{C}_{ij} of the form, with d_{ij} being analytic extension of \overline{d}_{ij} ,

$$\underline{C_{ij}} \equiv \delta_{ij} + \underline{c_{ij}}, \quad \underline{c_{ij}} \equiv y^0 \underline{d_{ij}},$$

therefore,

$$\underline{\partial_h C_{ij}} \equiv 0$$
 for $y^0 = 0$, hence $\underline{\partial_h C^{ij}} \equiv 0$ for $y^0 = 0$

and

$$\underline{C^{ij}} \equiv \delta^{ij} + y^0 \underline{d^{ij}},$$

with $\underline{d^{ij}}$ being some analytic functions. Hence

$$\underline{C^{ij}c_{ij}} \equiv (y^0)^2 \{ D + \underline{d^{ij}}\underline{d_{ij}} \}.$$

Using the definition of $\underline{X_{\alpha\beta}}$ we see that if C is near-round at the vertex, then

$$\underline{X_{\alpha\beta}} \equiv y^0 \underline{Y_{\alpha\beta}}, \quad \text{with} \quad \underline{Y_{ij}} := \frac{1}{2} \left\{ \underline{d_{ij}} + y^0 \underline{\partial_0 d_{ij}} + y^h \underline{\partial_h d_{ij}} \right\}, \quad \underline{Y_{i0}} \equiv \underline{Y_{00}} \equiv 0.$$
(6.17)

Hence

tr
$$X \equiv y^0$$
 tr Y , with tr $Y \equiv \underline{C^{ij}} \underline{Y_{ij}} \equiv \frac{1}{2} \underline{C^{ij}} \{ \underline{d_{ij}} + y^0 \underline{\partial_0 d_{ij}} + y^h \underline{\partial_h d_{ij}} \}.$

An elementary computation shows that $\operatorname{tr} Y$ is of the following form (with Z being an analytic function)

$$\operatorname{tr} Y \equiv y^0 Z$$
, hence $\operatorname{tr} X \equiv (y^0)^2 Z$.

On the other hand,

$$\underline{X_j^i} \equiv \underline{C^{ih}} \underline{X_{jh}} = y^0 \underline{C^{ih}} \underline{Y_{jh}} := y^0 \underline{Y_j^i},$$

therefore,

$$|X|^2 \equiv \underline{X_j^i X_i^j} \equiv (y^0)^2 \underline{Y_j^i Y_i^j}.$$

We deduce from these formulas that if C is near-round at the vertex then $r^{-2}\overline{\operatorname{tr} X} \equiv \overline{Z}$ and $r^{-2}|\overline{X}|^2 \equiv |\overline{Y}|^2$ are admissible series.

Theorem 6.2. A sufficient condition for $r^{-2}(\nu^0 - 1)$, with ν_0 solution of the first wave-map gauge constraint, to have an admissible expansion is that the C^{ω} metric C given by (3.3) which induces the basic characteristic data be near-round at the vertex.

Proof. Since
$$f := \nu_0 - 1$$
 satisfies Eq. (6.8), $\phi := r^{-2}(\nu_0 - 1)$ satisfies
 $r\partial_1 \phi + (2+k)\phi + r^{-2}h = 0.$ (6.18)

The expression (6.16) shows that for a metric
$$C$$
 round at the vertex $r^{-2}h$ admits an admissible development, the application of Corollary 5.1 gives the result.

7. The C_A constraint

We have written in [1] the C_A constraint in vacuum as

$$\mathcal{C}_A = -\frac{1}{2}(\partial_1\xi_A + \tau\xi_A) + \widetilde{\nabla}_B\chi^B_A - \frac{1}{2}\partial_A\tau + \partial_A\left(\frac{1}{2}\overline{W}_1 + \nu_0\partial_1\nu^0\right)$$

where ξ_A is defined as

$$\xi_A := -2\nu^0 \partial_1 \nu_A + 4\nu^0 \nu_C \chi_A^C + \left(\overline{W}^0 - \frac{2}{r}\nu^0\right) \nu_A + \overline{g}_{AB}\overline{g}^{CD} (S_{CD}^B - \widetilde{\Gamma}_{CD}^B).$$
(7.1)

Using the first constraint we find

$$\nu_0 \partial_1 \nu^0 + \frac{1}{2} \overline{W}_1 = -a, \qquad (7.2)$$

where a is given by (6.14), hence

$$\partial_A\left(a+\frac{1}{2}\tau\right) \equiv r^{-1}\partial_A F(\overline{\operatorname{tr} X}, |\overline{X}|^2),$$

where

$$F(\overline{\operatorname{tr} X}, |\overline{X}|^2) := \frac{r\partial_1 \overline{\operatorname{tr} X} + \overline{\operatorname{tr} X} + |\overline{X}|^2}{n - 1 + \overline{\operatorname{tr} X}} + \frac{1}{2} \overline{\operatorname{tr} X} \equiv \frac{r\partial_1 \overline{\operatorname{tr} X} + \frac{1}{2} \{(n + 1)\overline{\operatorname{tr} X} + |\overline{\operatorname{tr} X}|^2\} + |\overline{X}|^2}{n - 1 + \overline{\operatorname{tr} X}}$$

admits in a neighbourhood of ${\cal O}$ an admissible development of minimal order 2.

We have:

$$\mathcal{C}_A \equiv -\frac{1}{2} \left(\partial_1 \xi_A + \tau \xi_A \right) + \widetilde{\nabla}_B \chi_A^B - r^{-1} \partial_A F(\overline{\operatorname{tr} X}, |\overline{X}|^2) = 0.$$
(7.3)

7.1. Equations for ξ_A . We set $\xi_1 = \xi_0 = 0$ on the cone and we define $\underline{\xi_i}$ by

$$\underline{\xi_i} := \frac{\partial x^{\alpha}}{\partial y^i} \, \xi_{\alpha} \equiv \frac{\partial x^A}{\partial y^i} \, \xi_A. \tag{7.4}$$

It holds that

$$y^i \underline{\xi_i} = 0, \tag{7.5}$$

because (recall that $x^1 = r, \ y^i \equiv r \Theta^i(x^A))$

$$y^{i}\frac{\partial x^{A}}{\partial y^{i}} \equiv r\frac{\partial y^{i}}{\partial x^{1}}\frac{\partial x^{A}}{\partial y^{i}} \equiv r\delta_{1}^{A} = 0, \quad \frac{\partial}{\partial r}\frac{\partial y^{i}}{\partial x^{A}} \equiv \frac{1}{r}\frac{\partial y^{i}}{\partial x^{A}}.$$
(7.6)

We have

$$\frac{\partial y^i}{\partial x^A} \underline{\xi_i} \equiv \frac{\partial y^i}{\partial x^A} \frac{\partial x^B}{\partial y^i} \xi_B \equiv \delta_A^B \xi_B \equiv \xi_A. \tag{7.7}$$

The equation (7.7) implies that

$$\partial_1 \xi_A \equiv \left(\frac{\partial}{\partial r} \underline{\xi_i} + r^{-1} \underline{\xi_i}\right) \frac{\partial y^i}{\partial x^A},$$

hence

$$\mathcal{C}_A \equiv -\frac{1}{2} \frac{\partial y^i}{\partial x^A} \left\{ \frac{\partial}{\partial r} \underline{\xi_i} + \underline{\xi_i} (r^{-1} + \tau) \right\} + \widetilde{\nabla}_B \chi_A^B - r^{-1} \partial_A F(\overline{\operatorname{tr} X}, |\overline{X}|^2) = 0.$$

Since $\operatorname{tr} X \equiv \operatorname{tr} X$ is a scalar function and the equation of C_O in the x coordinates is $x^0 = 0$ and y^0 does not depend on x^A , it holds that

$$\frac{\partial}{\partial x^A} \overline{\operatorname{tr} X} \equiv \overline{\frac{\partial}{\partial x^A} \operatorname{tr} X} \equiv \overline{\frac{\partial y^i}{\partial x^A} \frac{\partial}{\partial y^i} \frac{\operatorname{tr} X}{\partial x^A}} \equiv \frac{\partial y^i}{\partial x^A} \overline{\frac{\partial}{\partial y^i} \frac{\operatorname{tr} X}{\partial x^A}},$$

analogously

$$\frac{\partial}{\partial x^A} \overline{|X|^2} \equiv \frac{\partial y^i}{\partial x^A} \overline{\frac{\partial}{\partial y^i} |X|^2}, \quad \frac{\partial}{\partial x^A} \overline{F(\operatorname{tr} X, |X|^2)} \equiv \frac{\partial y^i}{\partial x^A} \overline{\frac{\partial}{\partial y^i} F(\operatorname{tr} X, |X|^2)}.$$

We now compute, with covariant derivatives $\widetilde{\nabla}$ taken in the Riemannian metric $\overline{g}_{AB} \equiv \overline{C}_{AB}$,

$$\widetilde{\nabla}_B \chi^B_A \equiv \widetilde{\nabla}_B \left(\frac{1}{x^1} \overline{X}^B_A + \frac{1}{x^1} \delta^B_A \right) \equiv \frac{1}{x^1} \widetilde{\nabla}_B \overline{X}^B_A.$$

The Christoffel symbols \widetilde{C}^A_{BC} of the Riemannian connection $\widetilde{\nabla}$ are equal (recall that $C^{B0} = C^{B1} = C^{00} = 0$) to the trace on C_O of the Christoffel symbols with the same indices of the spacetime metric C, hence denoting by ${}^{(C)}\nabla$ the covariant derivative in the metric C

$$\widetilde{\nabla}_B \chi^B_A \equiv \frac{1}{x^1} \overline{{}^{(C)} \nabla_B X^B_A}.$$

Since the X_A^B are the only non-vanishing components of the tensor X, and due to the form chosen for the metric C we find that

$$\overline{{}^{(C)}\nabla_B X^B_A} = \overline{{}^{(C)}\nabla_\alpha X^\alpha_A} = \frac{\partial y^i}{\partial x^A} \underbrace{\overline{}^{(C)}\nabla_\alpha X^\alpha_i}$$

and Eqs. (7.3) read

$$\mathcal{C}_{A} \equiv \frac{\partial y^{j}}{\partial x^{A}} \frac{1}{r} \left\{ -\frac{1}{2} \left[r \frac{\partial}{\partial r} \underline{\xi_{j}} + \underline{\xi_{j}} (n + \overline{\operatorname{tr} X}) \right] + \underline{\overline{(C)}} \nabla_{\alpha} X_{j}^{\alpha} - \underline{\frac{\partial}{\partial y^{j}}} F(\operatorname{tr} X, |X|^{2}) \right\} = 0.$$
(7.8)

The parentheses constitute a linear diagonal operator on the $\underline{\xi}_j$ of the type considered in Lemma 5.3. Equating it to zero gives an equation with *solution* $\underline{\xi}_j$ an admissible series of minimal order 1. We denote by $\underline{\Xi}_j$ the extension of $\underline{\xi}_i$ to spacetime, that is we have

$$\underline{\xi_i} \equiv \overline{\underline{\Xi_i}},\tag{7.9}$$

where $\underline{\Xi_i}$ are analytical functions beginning by linear terms.

7.2. Equations for $\underline{\nu_i}$. We now consider Eqs. (7.1), which read

$$\partial_1 \nu_A + \left(\frac{1}{r} - \frac{1}{2}\overline{W}_1\right)\nu_A - 2\nu_C \chi_A^C - \frac{1}{2}\nu_0 \overline{g}_{AB}\overline{g}^{CD}(S_{CD}^B - \widetilde{\Gamma}_{CD}^B) + \frac{1}{2}\nu_0 \xi_A = 0.$$
(7.10)

We set

$$\underline{\overline{g}}_{0i} \equiv -\underline{\nu_i} + \lambda \underline{\overline{L}}_i, \quad \text{with} \quad \underline{\overline{L}}_i \equiv \underline{\overline{C}}_{ij} \underline{\overline{L}}^j, \quad \underline{\overline{L}}^j \equiv y^j, \tag{7.11}$$

with $\underline{\nu_i}$ such that

$$\underline{\nu_i \overline{L}^i} \equiv \underline{\nu_i} \, y^i = 0; \tag{7.12}$$

that is, using (7.11),

$$\lambda \equiv (\underline{\overline{L}_i \, \overline{L}^i})^{-1} \, \underline{\overline{g}_{0j} \, \overline{L}^j} \equiv r^{-2} y^j \, \underline{\overline{g}_{0j}}.$$

Then (compare (3.5))

$$\nu_A \equiv -\frac{\partial y^i}{\partial x^A} \, \overline{\underline{g}}_{0i} \equiv \frac{\partial y^i}{\partial x^A} \, \underline{\nu}_i.$$

Hence

$$\partial_1 \nu_A \equiv \frac{\partial y^i}{\partial x^A} (\partial_1 \underline{\nu_i} + r^{-1} \underline{\nu_i}).$$

We recall that

$$\chi_A^C \equiv \frac{1}{r} (\overline{X}_A^C + \delta_A^C), \quad \text{i.e.} \quad r\nu_C \chi_A^C \equiv \nu_C \overline{X}_A^C + \nu_A.$$

Therefore, after product by r, the equations can be written as follows:

$$\frac{\partial y^i}{\partial x^A} \left(r \partial_1 \underline{\nu_i} - \frac{1}{2} \, r \overline{W}_1 \underline{\nu_i} + \frac{1}{2} \, \nu_0 r \underline{\xi_i} \right) - 2 \overline{F}_A - \frac{1}{2} \, r \nu_0 \overline{E}_A = 0,$$

with

$$\overline{F}_A := \nu_C \overline{X}_A^C, \quad \overline{E}_A := \overline{g}_{AB} \overline{g}^{CD} (S_{CD}^B - \widetilde{\Gamma}_{CD}^B)$$

By definition it holds that

$$\nu_C \overline{X}_A^C \equiv \overline{g_{0C} X_A^C},$$

and since X_A^C are the only non-vanishing components of the mixed tensor X in the coordinates x

$$g_{0C}X_A^C = g_{0\lambda}X_A^{\lambda} \equiv \frac{\partial y^{\alpha}}{\partial x^0} \frac{\partial y^{\beta}}{\partial x^A} \underline{g_{\alpha\lambda}} \underline{X_{\beta}^{\lambda}}.$$

Recalling that $\underline{X_i^0} \equiv 0$ we find (using (4.11), $\underline{\overline{X_i^j}L_j} = 0$)

$$\overline{F}_A \equiv \nu_C \, \overline{X}_A^C \equiv -\frac{\partial y^i}{\partial x^A} \, \overline{g_{0j} X_i^j} \equiv \frac{\partial y^i}{\partial x^A} \, \overline{\nu_j X_i^j}.$$

We then remark that $S^B_{CD} - \tilde{C}^B_{CD}$ is the trace on C_O of the difference of the components of the Christoffel symbols, $\eta^{\alpha}_{\beta\gamma}$ and $C^{\alpha}_{\beta\gamma}$, with these angular x indices of the Minkowski metric η and the metric C:

$$\overline{E}_A := \overline{g}_{AB} \overline{g}^{CD} (S^B_{CD} - \widetilde{C}^B_{CD}), \quad \text{with} \quad E_A \equiv C_{AB} C^{CD} (\eta^B_{CD} - C^B_{CD}).$$

Using the expressions of η and C and the vanishing of the Christoffel symbols of η in y coordinates we find

$$\overline{E}_A \equiv \overline{C_{A\alpha}C^{\lambda\mu}(\eta^{\alpha}_{\lambda\mu} - C^{\alpha}_{\lambda\mu})} \equiv \frac{\partial y^i}{\partial x^A} \underbrace{\overline{C_{i\alpha}C^{\lambda\mu}(\eta^{\alpha}_{\lambda\mu} - C^{\alpha}_{\lambda\mu})}}_{= -\frac{\partial y^i}{\partial x^A}} \underbrace{\overline{C_{ij}C^{\lambda\mu}C^j_{\lambda\mu}}}_{= -\frac{\partial y^i}{\partial x^A}} \underbrace{\overline{C_{ij}C^{\lambda\mu}C^j_{\mu}}}_{= -\frac{\partial y^i}{\partial x^A}} \underbrace{\overline{C_{ij}C^j_{\mu}}}_{= -\frac{$$

with $C_{\lambda\mu}^{j}$ analytic functions, components of Christoffel symbols of the metric C in y coordinates, that is, using the values of the y components of the metric C

$$\underline{C_{ij}C^{\lambda\mu}C_{\lambda\mu}^{j}} \equiv \frac{1}{2} \underline{C^{hk}(\partial_h c_{ik} + \partial_h c_{ik} - \partial_i c_{hk})}.$$

We recall from (6.15) that

$$r\overline{W}_1 \equiv -\nu_0 \,\overline{g}^{AB} r^2 s_{AB} \equiv -\nu_0 \,\{n-1 + \overline{c^{ij} \delta_{ij}}\},$$

and we find that Eqs. (7.10) can be written as

$$\frac{\partial y^i}{\partial x^A} \mathcal{L}_i = 0$$

where \mathcal{L}_i is the following linear operator on $\underline{\nu_i}$

$$r\partial_1\underline{\nu_i} + \nu_0 \left\{ \frac{n-1}{2} + \frac{1}{2} \overline{\underline{c^{hk}\delta_{hk}}} \right\} \nu_i - 2\overline{\underline{\nu_j X_i^j}} + \frac{1}{2}\nu_0 r \underline{\xi_i} - \frac{1}{2} r\nu_0 \overline{\underline{C_{ij} C^{\lambda\mu} C_{\lambda\mu}^j}} = 0.$$
(7.13)

We extend Lemma 5.3 as follows.

Lemma 7.1. If k_i^j and h_i are admissible series of minimal orders 1, and the constant $k_0 \ge 0$, then the ODE

$$r\partial_1 \underline{\nu_i} + k_0 \underline{\nu_i} + k_i^j \underline{\nu_j} = h_i \tag{7.14}$$

admits a solution $\underline{\nu_i}$, which is also an admissible series of the same minimal order than h.

Recalling that $\nu_0 - 1$ is an admissible series of minimal order 2 we see that this lemma applies to (7.13). We have proved:

Theorem 7.1. If the basic characteristic data is induced by an analytic metric C satisfying (3.3), (3.4), Eq. (7.13) admits one and only one solution $\underline{\nu_i}$ which is an admissible series of minimal order 2. We denote by $\underline{N_i}$ its spacetime extension.

We now prove:

Theorem 7.2. A sufficient condition for $r^{-2}\underline{\nu_i}$ to have an admissible expansion is that the C^{ω} metric C given by (3.3) which induces the basic characteristic data be near-round at the vertex.

Proof. Using the relation between X and Y the linearity of F in trX and $|X|^2$ we see that for C round at the vertex the equation satisfied by ξ_j reads

$$\frac{1}{2}\left\{r\frac{\partial}{\partial r}\underline{\xi_j} + \underline{\xi_j}(n+r^2\overline{Z})\right\} = h_j,$$

with

$$h_j := \overline{(C)\nabla_{\alpha}((y^0)^2 Y_j^{\alpha})} - \overline{\frac{\partial}{\partial y^j}} F((y^0)^2 \operatorname{tr} Y, (y^0)^4 |Y|^2).$$

We deduce from the linearity of F in tr X and $|X|^2$ that $r^{-1}h_j$ admits an admissible expansion, the same holds therefore (see Corollary 5.1) for $r^{-1}\underline{\xi}_j$. The equation satisfied by $\underline{\nu}_j$ reads

$$r\partial_{1}\underline{\nu_{i}} + \nu_{0}\left\{\frac{n-1}{2} + \frac{r^{2}d^{hk}\delta_{hk}}{2}\right\}\underline{\nu_{i}} - 2\nu_{j}r^{2}\overline{Y_{i}^{j}} = h_{i},$$

$$h_{i} = -\frac{1}{2}\nu_{0}r\underline{\xi_{i}} + \frac{1}{4}r^{3}\nu_{0}\overline{C^{hk}(\partial_{h}d_{ik} + \partial_{h}d_{ik} - \partial_{i}d_{hk})}.$$

$$(7.15)$$

An extension of Lemma 7.1 shows that $r^{-2}\underline{\nu_i}$ admits an admissible expansion because it is so of h_i .

8. The C_0 constraint

The last unknown in \overline{g} , the only unknown in the constraint \mathcal{C}_0 , is

$$\overline{g}_{00} \equiv \overline{g}_{00}$$

The constraint C_0 has a simpler expression in terms of \overline{g}^{11} . Since \overline{g}^{11} is linked to \overline{g}_{00} by the identity

$$\overline{g}^{01}\overline{g}_{00} + \overline{g}^{11}\overline{g}_{10} + \overline{g}^{A1}\overline{g}_{A0} = 0,$$

we have

$$\overline{g}_{00} \equiv -\overline{g}^{11}(\nu_0)^2 + \overline{g}^{AB}\nu_B\nu_A \equiv -\overline{g}^{11}(\nu_0)^2 + \underline{\overline{C}}^{ij}\underline{\nu_i}\underline{\nu_j}.$$
(8.1)

We have seen [1] that the C_0 constraint can be written in vacuum as

$$\partial_1 \zeta + (\kappa + \tau) \zeta + \frac{1}{2} \left\{ \partial_1 \overline{W}^1 + (\kappa + \tau) \overline{W}^1 + \widetilde{R} - \frac{1}{2} \overline{g}^{AB} \xi_A \xi_B + \overline{g}^{AB} \widetilde{\nabla}_A \xi_B \right\} = 0, \quad (8.2)$$

with

$$\zeta := \left(\partial_1 + \kappa + \frac{1}{2}\tau\right)\overline{g}^{11} + \frac{1}{2}\overline{W}^1,\tag{8.3}$$

$$\kappa \equiv \nu^0 \partial_1 \nu_0 - \frac{1}{2} (\overline{W}_1 + \tau), \quad \overline{W}_1 \equiv \nu_0 \overline{W}^0, \quad \overline{W}^0 \equiv \overline{W}^1 \equiv -r \overline{g}^{AB} s_{AB}.$$
(8.4)

8.1. Equation for ζ . In the flat case it holds that

$$\nu_{0,\eta} = 1, \quad \tau_{\eta} = -\overline{W}_{1,\eta} = \frac{n-1}{r}, \quad \kappa_{\eta} = 0.$$

The function $\zeta\,$ reduces to

$$\zeta_{\eta} := \partial_1 \overline{g}^{11} + \frac{1}{2} \tau_{\eta} (\overline{g}^{11} - 1), \tag{8.5}$$

and the equation for ζ reads, using $\xi_{A,\eta} = 0$,

$$\partial_1 \zeta_\eta + \frac{n-1}{r} \zeta_\eta + \frac{1}{2} \left\{ \frac{n-1}{r^2} - \frac{(n-1)^2}{r^2} + \widetilde{R}_\eta \right\} = 0.$$

That is, using the scalar curvature of the S^{n-1} round sphere of radius r which is⁵

$$\widetilde{R}_{\eta} = r^{-2}(n-2)(n-1),$$
(8.6)

the equation

$$\partial_1 \zeta_\eta + \frac{n-1}{r} \, \zeta_\eta = 0$$

has the only bounded solution $\zeta_{\eta} \equiv 0$. From (8.5) results we have then $\overline{g}_{\eta}^{11} \equiv 1$. We now study the general case.

We can write the equation to be satisfied by ζ as follows,

$$r\partial_1(r\zeta) + {}^{(r\zeta)}k\,r\zeta + {}^{(r\zeta)}h = 0, \qquad (8.7)$$

$${}^{(r\zeta)}k := r(\kappa + \tau) - 1 \equiv r \left\{ \nu^0 \partial_1 \nu_0 + \frac{1}{2}(\tau - \overline{W_1}) \right\} - 1, \tag{8.8}$$

$${}^{(r\zeta)}h := \frac{r^2}{2} \left\{ \partial_1 \overline{W}^1 + (\kappa + \tau) \overline{W}^1 + \widetilde{R} - \frac{1}{2} \overline{g}^{AB} \xi_A \xi_B + \overline{g}^{AB} \widetilde{\nabla}_A \xi_B \right\}.$$

Hence

$${}^{(r\zeta)}h := \frac{r^2}{2} \left\{ \partial_1 \overline{W}^1 + \frac{1}{2} (\tau - \overline{W}_1) \overline{W}^1 + \widetilde{R} + \nu^0 \partial_1 \nu_0 \overline{W}^1 - \frac{1}{2} \overline{g}^{AB} \xi_A \xi_B + \overline{g}^{AB} \widetilde{\nabla}_A \xi_B \right\}.$$
(8.9)

We have shown that $\nu^0 - 1$ and $r\partial_1\nu_0$, hence also $r\nu^0\partial_1\nu_0$, admit admissible expansions of minimal order 2, and we have seen that $r\tau$ and $r\overline{W_1}$ are admissible series with terms of order zero (n-1) and -(n-1) respectively. Hence k is an admissible series of the

⁵See for instance [6, p. 140].

zero-order term n-2, and we have

$${}^{(r\zeta)}k = n - 2 + k_1,$$

where k_1 is an admissible series of minimal order 1.

We study the terms appearing in $(r\zeta)h$.

The term

$$r^2 \nu^0 \partial_1 \nu_0 \overline{W}^1 \equiv r \nu^0 \partial_1 \nu_0 \, r \overline{W}^1 \tag{8.10}$$

has an admissible expansion of minimal order 2 (see Lemma 5.4).

The term

$$r^2 \overline{g}^{AB} \xi_A \xi_B \equiv r^2 \overline{\underline{C}}^{ij} \underline{\xi}_i \underline{\xi}_j \tag{8.11}$$

has an admissible expansion of order 4 because $\underline{\xi_i}$ has an admissible expansion of order 1. As for the term

$$\overline{g}^{AB}\widetilde{\nabla}_A\xi_B \equiv \overline{C}^{AB}\widetilde{\nabla}_A\xi_B,$$

the Christoffel symbols \widetilde{C}_{BC}^A of the Riemannian connection $\widetilde{\nabla}$ are equal (recall that $C^{B0} = C^{B1} = C^{00} = 0$) to the trace on C_O of the Christoffel symbols with the same indices of the spacetime metric C, hence denoting by ${}^{(C)}\nabla$ the covariant derivative in the metric C

$$\widetilde{\nabla}_A \xi_B \equiv \overline{{}^{(C)} \nabla_A \Xi_B}.$$

Since the Ξ_B are the only non-vanishing components of the vector Ξ , and due to the form chosen for the metric C we find

$$\overline{C}^{AB}\overline{(C)}\nabla_A\Xi_B = \overline{C^{\alpha\beta} (C)}\nabla_{\alpha}\Xi_{\beta}.$$
(8.12)

Hence the scalar $r^2 \overline{C}^{AB} \widetilde{\nabla}_A \xi_B$ has an admissible expansion of minimal order 2. We have seen that in the flat case

$$\overline{W}^{1}_{\eta} \equiv \eta^{\alpha\beta} \,\widehat{\Gamma}^{1}_{\alpha\beta} \equiv -\frac{n-1}{r} \tag{8.13}$$

and

$$\partial_1 \overline{W}^1_{\eta} + \tau \overline{W}^1_{\eta} \equiv \left\{ \frac{n-1}{r^2} - \frac{(n-1)^2}{r^2} \right\} = -\widetilde{R}_{\eta} .$$
(8.14)

In the general case we compute

$$\partial_1 \overline{W}^1 + \frac{1}{2} (\tau - \overline{W}_1) \overline{W}^1.$$

Recall that

$$\tau \equiv \frac{n-1}{r} + \frac{\overline{\operatorname{tr} X}}{r};$$

 set

$$\overline{W}^1 \equiv \overline{W}^1_{\eta} + F, \quad \text{with} \quad F := (\overline{g}^{\alpha\beta} - \eta^{\alpha\beta}) \,\widehat{\Gamma}^1_{\alpha\beta}.$$

Using the values of the Christoffel symbols $\hat{\Gamma}^1_{\alpha\beta}$ and the components of g and η in x coordinates we find

$$F \equiv -(\overline{C}^{AB} - \eta^{AB})x^{1}s_{AB} \equiv -\frac{1}{r}(\overline{C}^{\alpha\beta} - \eta^{\alpha\beta})\eta_{\alpha\beta} \equiv \\ \equiv \frac{1}{r}(n+1 - \overline{C}^{\alpha\beta}\eta_{\alpha\beta}) \equiv \frac{1}{r}(n+1 - \overline{C}^{\alpha\beta}\eta_{\alpha\beta}) \equiv \frac{1}{r}(n - \overline{C}^{ij}\eta_{ij}) \equiv -\frac{1}{r}\overline{\underline{c}}^{ij}\delta_{ij};$$

,

hence

$$\overline{W}^{1} \equiv -\frac{n-1}{r} - \frac{\overline{\underline{c}}^{ij}\delta_{ij}}{r}, \quad \partial_{1}\overline{W}^{1} \equiv \frac{n-1}{r^{2}} + \frac{(\overline{\underline{c}}^{ij} - y^{h}\underline{\partial}_{h}\overline{c}^{ij})\delta_{ij}}{r^{2}}$$

and

$$\frac{1}{2}(\tau - \overline{W}^1)\overline{W}^1 \equiv -\left\{\frac{n-1}{r} + \frac{\overline{\operatorname{tr} X} + \overline{\underline{c}^{ij}}\delta_{ij}}{2r}\right\} \left\{\frac{n-1}{r} + \frac{1}{r}\overline{\underline{c}^{ij}}\delta_{ij}\right\}.$$

Using the value of the scalar curvature \widetilde{R}_{η} of the round sphere S^{n-1} of radius r we find that

$$r^{2}\left\{\partial_{1}\overline{W}^{1} + \frac{1}{2}(\tau - \overline{W}^{1})\overline{W}^{1}\right\} \equiv -r^{2}\widetilde{R}_{\eta} + \Phi,$$

where Φ is an admissible expansion of minimal order 2,

$$\Phi := -y^{h} \underline{\partial_{h} \overline{c}^{ij}} \delta_{ij} - (n-2) \underline{\overline{c}^{ij}} \delta_{ij} - \frac{1}{2} (n-1 + \underline{\overline{c}^{hk}} \delta_{hk}) (\overline{\operatorname{tr} X} + \underline{\overline{c}^{ij}} \delta_{ij}).$$

To compute $r^2 \widetilde{R}$, we use formulas given in [1]. The formulas (10.33) and (10.37) of [1] for a general metric in null adapted coordinates are

$$\begin{split} \overline{g}^{AB}\overline{R}_{AB} &\equiv 2(\partial_1 + \overline{\Gamma}^1_{11} + \tau) \left[(\partial_1 + \overline{\Gamma}^1_{11} + \frac{\tau}{2})\overline{g}^{11} + \overline{\Gamma}^1 \right] + \widetilde{R} - 2\overline{g}^{AB}\overline{\Gamma}^1_{1A}\overline{\Gamma}^1_{1B} - 2\overline{g}^{AB}\widetilde{\nabla}_A\overline{\Gamma}^1_{1B}, \\ \overline{R}_{11} &\equiv -\partial_1\tau + \overline{\Gamma}^1_{11}\tau - \chi^B_A\chi^A_B, \end{split}$$

and

$$\overline{S}_{01} \equiv -\frac{1}{2}\nu_0 \overline{g}^{AB} \overline{R}_{AB} + \overline{R}_{1A}\nu^A - \frac{1}{2}\nu_0 \overline{g}^{11} \overline{R}_{11}$$

In the case of the metric C, it holds that $C_{01} = 1$, $C_{0A} = 0$, $C_{00} = -1$, and $C^{11} = 1$. Hence

$${}^{(C)}\Gamma^{1}_{11} \equiv {}^{(C)}\Gamma^{1}_{1A} \equiv 0, \quad {}^{(C)}\Gamma^{1} \equiv -\frac{1}{2}(C^{AB}\partial_{1}C_{AB} + C^{AB}\partial_{0}C_{AB}),$$

and the above formulas reduce to (recall that $\widetilde{R} \equiv {}^{(C)}\widetilde{R}$):

$$\overline{C}^{AB} \ ^{(C)}\overline{R}_{AB} \equiv -(\partial_1 + \tau) \left[\tau + \overline{C}^{AB} \overline{\partial_0 C_{AB}}\right] + \widetilde{R},$$
$$\overline{R}_{11} \equiv -\partial_1 \tau - \chi^B_A \chi^A_B,$$

 and

$$-2 \ ^{(C)}\overline{S}_{01} \equiv \overline{g}^{AB}\overline{R}_{AB} + \overline{R}_{11},$$

from which we deduce

$$\widetilde{R} \equiv 2\partial_1 \tau + \tau^2 + (\partial_1 + \tau)\overline{C}^{AB}\overline{\partial_0 C_{AB}} + \chi^B_A \chi^A_B - 2 \ ^{(C)}\overline{S}_{01}.$$

We have

$${}^{(C)}\overline{S}_{01} \equiv -\left({}^{(C)}\underline{\overline{S}_{00}} + r^{-1}y^{i} {}^{(C)}\underline{\overline{S}_{0i}}\right).$$

Since C is an analytic metric in a neighbourhood of O, $(C)\overline{S}_{\alpha\beta}$ admit admissible expansions and, hence $r^{2} {}^{(C)}\overline{S}_{01}$ also admits an admissible expansion of minimal order 2. Recall that

$$\tau \equiv \frac{n-1}{r} + \frac{\overline{\operatorname{tr} X}}{r} \quad \text{and} \quad |\chi|^2 = \frac{|\overline{X}|^2 + 2\overline{\operatorname{tr} X} + n - 1}{r^2},$$

hence

$$r^{2} \{ 2\partial_{1}\tau + \tau^{2} + \chi^{B}_{A}\chi^{A}_{B} \} \equiv (n-1)(n-2) + 2(n-1)\overline{\operatorname{tr} X} + 2r\partial_{1}\overline{\operatorname{tr} X} + (\overline{\operatorname{tr} X})^{2} + |\overline{X}|^{2}.$$

Finally we remark that $\partial_0 C_{AB}$ are the non-vanishing components of the Lie derivative of the metric C with respect to the vector m with x components $m^0 = 1$, $m^1 = m^A = 0$, hence with y components $\underline{m}^0 = -1$, $\underline{m}^i = 0$ that

$$C^{AB}\partial_0 C_{AB} \equiv C^{\alpha\beta}\mathcal{L}_m C_{\alpha\beta} \equiv \underline{C^{\alpha\beta}\partial_0 C_{\alpha\beta}},$$

hence

$$\overline{C}^{AB}\overline{\partial_0 C_{AB}} \equiv \overline{\underline{C^{ij}}} \, \overline{\partial_0 C_{ij}} \equiv \overline{\underline{C^{ij}}} \, \overline{\partial_0 c_{ij}}$$

has an admissible expansion of minimal order 1 and $r^2(\partial_1 + \tau)\overline{C}^{AB}\overline{\partial_0 C_{AB}}$ has an admissible expansion of minimal order 2.

We have proved that

$$r^2 \widetilde{R} \equiv r^2 \widetilde{R}_\eta + \Psi,$$

where

$$\Psi \equiv 2(n-1)\overline{\operatorname{tr} X} + 2r\partial_1\overline{\operatorname{tr} X} + (\overline{\operatorname{tr} X})^2 + |\overline{X}|^2 + r^2(\partial_1 + \tau)\overline{\underline{C^{ij}}}\,\underline{\partial_0 c_{ij}}$$

has an admissible expansion of minimal order 2. Hence

$$r^{2}\left\{\partial_{1}\overline{W}^{1} + \frac{1}{2}(\tau - \overline{W}^{1})\overline{W}^{1} + \widetilde{R}\right\} \equiv \Phi + \Psi.$$

In conclusion, we have shown that ${}^{(r\zeta)}h$ has an admissible expansion of minimal order 2, the same is therefore true for $r\zeta$.

8.2. Equation for $\alpha := \overline{g}^{11} - 1$. The definition of ζ gives for $\alpha := \overline{g}^{11} - 1$ an equation which reads

$$r\partial_1 \alpha + r\left(\kappa + \frac{1}{2}\tau\right)\alpha + \frac{r}{2}(2\kappa + \tau + \overline{W}_1 - 2\zeta) = 0.$$
(8.15)

Theorem 8.1. If the basic characteristic data is induced in a neighbourhood of O by an analytic metric C satisfying (3.3), (3.4), then Eq. (8.15) admits in this neighbourhood one and only one solution α which is an admissible series of minimal order 2.

This implies that $\overline{g}_{00} + 1$ is also an admissible series of minimal order 2.

Proof. Using the definition of κ in (8.4), the equation (8.15) reads

$$r\partial_1 \alpha + r\left(\kappa + \frac{1}{2}\tau\right) \alpha + r(\nu^0 \partial_1 \nu_0 - \zeta) = 0,$$

that is,

$$r\partial_1 \alpha + {}^{(\alpha)}k\alpha + {}^{(\alpha)}h = 0, \qquad (8.16)$$

with

$${}^{(\alpha)}k := r\left(\nu^0\partial_1\nu_0 - \frac{1}{2}\overline{W}_1\right), \quad {}^{(\alpha)}h := r\left(\nu^0\partial_1\nu_0 - \zeta\right). \tag{8.17}$$

Previous results show that this equation is of a form to which Lemma 5.3 applies with ${}^{(\alpha)}k_0 = \frac{n-1}{2}$ and ${}^{(\alpha)}h$ of minimal order 2. It has therefore a solution α , admissible series of minimal order 2.

The identity (8.1) shows the property of $\overline{g}_{00} + 1$.

Theorem 8.2. If in addition to the hypothesis of the previous theorem the given metric C is near-round at the vertex, then $r^{-2}(\overline{g}_{00} + 1)$ is an admissible series in a neighbourhood of the vertex.

Proof. The result will follow from the proof that ${}^{(\alpha)}h$ given in (8.17) is such that $r^{-2(\alpha)}h$ is an admissible series. By previous results, it remains only to prove that $r^{-2}(r\zeta)$ has an admissible expansion, hence that it is so for $r^{-2(r\zeta)}h$. We have

$$r^{-2(r\zeta)}h := \frac{1}{2} \left\{ r^{-2}(\Phi + \Psi) + \nu^0 \partial_1 \nu_0 \overline{W}^1 - \frac{1}{2} \overline{g}^{AB} \xi_A \xi_B + \overline{g}^{AB} \widetilde{\nabla}_A \xi_B \right\}.$$
 (8.18)

We check that if C is near-round at the vertex, the various terms studied above are such that the required condition is satisfied. Indeed,

$$\nu^0 \partial_1 \nu_0 \overline{W}^1 \equiv \nu^0 r^{-1} \partial_1 \nu_0 r \overline{W}^1$$

has an admissible expansion because it is so for $r^{-1}\partial_1\nu_0$ and $r\overline{W}^1$.

$$\overline{g}^{AB}\xi_A\xi_B \equiv \overline{\underline{C}}^{ij}\underline{\xi_i}\underline{\xi_j}, \quad \overline{g}^{AB}\widetilde{\nabla}_A\xi_B \equiv \overline{\underline{C}^{\alpha\beta}(C)}\nabla_\alpha\Xi_\beta$$

have admissible expansions because $\underline{\xi_i}$ does.

The assumptions on $\underline{c^{ij}}$ and the identity $y^h \partial_h r \equiv r$ show that

$$y^{h}\partial_{h}\overline{\underline{c}^{ij}\delta_{ij}} \equiv r^{2}(y^{h}\partial_{h}\overline{\underline{d}}^{ij}\delta_{ij} + 2\overline{\underline{d}}^{ij}\delta_{ij}),$$

hence $r^{-2}\Phi$, and $r^{-2}\Psi$, with Φ and Ψ given above have admissible expansions.

It remains to show the property for

$${}^{(C)}\overline{S}_{01} \equiv -({}^{(C)}\underline{\overline{S}_{00}} + r^{-1}y^{i} {}^{(C)}\underline{\overline{S}_{0i}})$$

Since *C* is an analytic metric in a neighbourhood of *O*, ${}^{(C)}\overline{\underline{S}}_{00}$ has an admissible expansion. Denoting by ${}^{(C)}\underline{K}_{ij} \equiv -\frac{1}{2} \underline{\partial_0 C_{ij}}$ the second fundamental form of *C* relative to the slicing of R^{n+1} by $y^0 = \text{constant}$, we know that⁶,

$${}^{(C)}\overline{\underline{S}}_{0i} \equiv \underline{\partial}_i \operatorname{tr}^{(C)} K - {}^{(C)} \overline{\nabla}_j (C^{jh} {}^{(C)} K_{ih}).$$

We have

$$(C)\overline{\nabla_j(C^{jh(C)}K_{ih})} \equiv \underline{C^{jh}}\left(\partial_j (C)K_{ih} - C_{ji}^{k(C)}K_{ik} - C_{jh}^{k(C)}K_{jk}\right)$$

The functions ${}^{(C)}K_{ij}$ are analytic and the Christoffel symbols $\underline{C_{ji}^k}$ are products by y^0 of analytic functions, while elementary computations give

$$-2y^{i}\underline{C^{jh}}\underline{\partial_{j}}^{(C)}K_{ih} \equiv y^{i}\underline{C^{jh}}\underline{\partial_{j}\partial_{0}C_{ih}} \equiv \underline{C^{jh}}\left\{\underline{\partial_{j}\partial_{0}(y^{i}c_{ih})} - \underline{\partial_{0}c_{jh}}\right\}$$

hence using $y^i \underline{c_{ih}} = 0$ and $\delta^{jh} \underline{c_{jh}} = (y^0)^2 Z$. We deduce from these results that $r^{-1} y^{i(C)} \overline{\underline{S}}_{0i}$ also has an admissible expansion.

⁶See for instance [7, Chapter 6].

9. Conclusions

9.1. An existence theorem. We have shown that when the metric C given by (3.3) which induces on C_O the basic characteristic data \tilde{g} (i.e. $\overline{g}_{AB} \equiv \overline{C}_{AB} \equiv \widetilde{C}_{AB}$) is analytic, then the functions ν_0 , $\underline{\nu_i}$, \overline{g}_{00} have admissible expansions. We have shown that if moreover C is near-round at the vertex (definition 6.2), then the functions $r^{-2}(\nu_0 - 1)$, $r^{-1}\underline{\nu_i}$, and $r^{-2}(\overline{g}_{00} + 1)$ have also admissible expansions. These results imply the following theorem (the notations are those of Section 1):

Theorem 9.1. If the metric C given by (3.3), (3.4) which induces the basic characteristic data on the cone C_O^T is smooth everywhere, and moreover analytic and nearround in a neighbourhood of the vertex, then there exists a number $T_0 > 0$ such that the wave-gauge reduced vacuum Einstein equations with characteristic initial determined by C and the solution of the wave-map gauge constraints have a solution in $Y_O^{T_0}$ which induces on $C_O^{T_0}$ the same quadratic form as C.

Proof. It results from the formulae

$$\underline{\overline{g}}_{00} \equiv \overline{g}_{00}, \quad \underline{\overline{g}}_{0i} \equiv -\{\overline{g}_{00} + \nu_0\} r^{-1} y^i - \underline{\nu_i}, \quad \text{with} \quad y^i \underline{\nu_i} \equiv 0, \tag{9.1}$$

$$\overline{g}_{ij} - \delta_{ij} \equiv \{\overline{g}_{00} + 1 + 2(\nu_0 - 1)\}r^{-2}y^i y^j + r^{-1}(y^i \underline{\nu_j} + y^j \underline{\nu_i}) + \underline{c_{ij}},$$
(9.2)

and the theorems of previous sections that $\overline{g}_{00} + 1$, \overline{g}_{0i} , and $\overline{g}_{ij} - \delta_{ij}$ have admissible expansions, hence are the trace on C_O of analytic functions. We apply the Cagnac–Dossa theorem.

This theorem and the results of [1] lead then to the following sufficient conditions for the existence of a solution of the full Einstein equations.

Theorem 9.2. If the metric C given by (3.3) which induces the basic characteristic data on the cone C_O is smooth, and analytic and near-round at the vertex, there exists a vacuum Einsteinian spacetime $(Y_O^{T_0}, g)$ which induces on $C_O^{T_0}$ the same quadratic form as C. The solution is locally geometrically unique.

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Резюме

И. Шоке-Брюа, П.Т. Хрусьцель, Х.М. Мартин-Гарсиа. Теорема существования решения задачи Коши о световом конусе для вакуумных уравнений Эйнштейна с «почтизакругленными» аналитическими данными.

Представлен класс характеристических релятивистски обобщенных исходных данных, удовлетворяющих условию *почтизакругленности* на вершине светового конуса. Показано, что для любых подобных аналитических данных существует соответствующее решение для вакуумных уравнений Эйнштейна, определенное в окрестности вершины в световом конусе будущего.

Ключевые слова: характеристическая задача Коши о световом конусе, вакуумные уравнения Эйнштейна.

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