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PROBLEM OF SYNTHESIS OF OPTIMAL SYSTEMS

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In the theory of control the problem of synthesis of optimal systems (optimal control by the feedback principle) is one of the basic problems. A constructive solution of this problem makes it possible not only to solve efficiently the applied problems of optimization of dynamical systems, but also investigate a series of problems which are extremal by their statement (stabilization, decrement, amortization, etc.). Following [1] (Chap. 1, p. 16), in realizing a control under permanent perturbations the following three principles of control are used: 1) the principle of the feedback, 2) that of the direct coupling, 3) that of the direct-feedback coupling (combined principle). Principle 1), which uses in the formation of the controls the output signals of the system, is more universal than 2). However, situations exist in which it is advisable to apply the second principle based on available measurements of perturbations (input signals). The third combined principle of control is based on both 1) and 2) and can be realized as the direct and feedback couplings (it uses available measurements of both the input and output signals).

In this article for linear nonstationary dynamical systems we suggest an approach to the problem of synthesis of optimal systems which function in the conditions of permanent perturbations. This approach is based on the procedure of corrections of optimal program controls and rapid optimization algorithms. In the class of discrete controls we substantiate the algorithms of construction of realizations of optimal discrete feedback, direct couplings, and combined couplings. In Section 1 we expose the approach to the construction of optimal closable (see [2], Chap. 1, p. 18; [3]) and closed inverse couplings, which ensure optimal guaranteed results for control systems which are subject to the action of bounded perturbations of a non-stochastic nature. In Section 2 we study the optimal breakable couplings which are constructed by the results of measurements of perturbations (the principle of direct coupling). In Section 3, in constructing a realization of optimal couplings, we apply the combined principle of control, which uses both the input and output signals.

1. Synthesis of optimal feedback couplings for dynamical systems with perturbations

Let $T = [t_*, t^*]$ be an interval of control, $h = (t^* - t_*)/N$ the period of time quantifying, N a natural number, $T_h = \{t_*,\ t_*+h,\ t_*+(N-1)h\}$. A function $u(t)$, $t \in T$, is called a discrete control if

$$u(t) = u(t_* + kh), \quad t \in [t_* + kh, t_* + (k+1)h[, \quad k = \overline{0, N-1}.$$

In the class of the discrete controls we consider the optimal control problem

$$\begin{gathered} c'x(t^*) \to \max, \quad \dot{x} = A(t)x + b(t)u + d(t)w, \quad x(t_*) = x_0, \\ x(t^*) \in X^* = \{x \in R^n : g_* \le Hx \le g^*\}, \quad u(t) \in U = \{u \in R : |u| \le 1\}, \\ w(t) \in W = \{w \in R : |w| \le 1\}, \quad t \in T, \end{gathered} \tag{1.1}$$

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The algorithm of solving problems of the type (1.13) was described in [5]. If it turns out that $\overline{X}_{\gamma^*}^{*,\alpha} = \emptyset$ for all sufficiently small α, then we increase the value γ^*. We define the set

$$X_{\gamma^p}^{p,\alpha} = \{x \in R^n : f_j' x \geq \beta_j^\alpha(t^p) - \gamma^p,\ j = \overline{1,q}\},$$

by giving a rather small initial value $\gamma^p > 0$. Let us pass to the segment $[t^{p-1}, t^p]$. For every vector from (1.11) we solve the problem

$$\max_{z,u} f_j' z = f_j' x_j^{p-1,\alpha} = \beta_j^\alpha(t^{p-1}),$$
$$\dot{x} = A(t)x + b(t)u, \quad x(t^{p-1}) = z,$$
$$x(t^p) \in \overline{X}_{\gamma^p}^{p,\alpha} = \{x \in R^n : f_j' x \geq \beta_j^\alpha(t^p) - \gamma^p - \alpha_j^p,\ j = \overline{1,q}\}, \quad u(t) \in U, \quad t \in [t^{p-1}, t^p],$$

where the numbers α_j^p, $j = \overline{1,q}$, are calculated by rules which are same for the numbers α_0, $\underline{\alpha_l^*}$, $\overline{\alpha_l^*}$, $l = \overline{1,m}$. If $\overline{X}_{\gamma^p}^{p,\alpha} = \emptyset$ for sufficiently small α, then we increase γ^p. By continuing this process, we construct the set $\overline{X}_{\gamma^1}^{1,\alpha} \neq \emptyset$. Let us find the optimal program control $\overline{u}^0(t)$, $t \in T$, for the state $x(t_*) = x_0$. To this end we first solve the problem

$$\gamma^1 \to \min, \quad \dot{x} = A(t)x + b(t)u, \quad x(t^*) = x_0, \tag{1.14}$$
$$x(t^1) \in \overline{X}_{\gamma^1}^{1,\alpha}, \quad u(t) \in U, \quad t \in [t_*, t^1[.$$

If in problem (1.14) no admissible solutions are present, then we increase γ^1. After that we find $\overline{u}^0(t)$, $t \in [t_*, t^1[$, by solving the problem

$$\alpha(t^1) = \max \alpha, \quad \dot{x} = A(t)x + b(t)u, \quad x(t_*) = x_0,$$
$$x(t^1) \in \overline{X}_{\gamma^1}^{1,\alpha}, \quad u(t) \in U, \quad t \in [t^*, t^1[.$$

Let $\overline{x}^0(t^1)$ be a state of the deterministic system (1.4), $x(t_*) = x_0$, at the moment t^1 after action of the control $\overline{u}^0(t)$, $t \in [t_*, t^1[$. Consider the problem

$$\gamma^2 \to \min, \quad \dot{x} = A(t)x + b(t)u, \quad x(t^1) = \overline{x}^0(t^1), \tag{1.15}$$
$$x(t^1) \in \overline{X}_{\gamma^2}^{2,\alpha}, \quad u(t) \in U, \quad t \in [t_1, t^2[.$$

If the constraints of problem (1.15) are contradictory, we increase γ^2. Afterwards we find the control $\overline{u}^0(t)$, $t \in [t^1, t^2[$, by solving the problem

$$\alpha(t^2) = \max \alpha, \quad \dot{x} = A(t)x + b(t)u, \quad x(t_1) = \overline{x}^0(t^1),$$
$$x(t^2) \in \overline{X}_{\gamma^2}^{2,\alpha}, \quad u(t) \in U, \quad t \in [t^1, t^2[.$$

By continuing the process, we construct $\overline{u}^0(t)$, $t \in T$.

1.2. *Procedure of correction.* Let us clarify the procedure only for the first segment. Consider the segment $[t_*, t^1[$. We denote by J^1 the set of indices which are active on the vector $\overline{x}^0(t^1)$ of constraints at the moment t^1. Let us construct the vectors $\Delta f_j^1 = \overline{x}^0(t^1) - x_j^{1,\alpha}$, $j \in J^1$, and put

$$\widetilde{f}_j^1 = f_j + \theta \Delta f_j^1, \quad j \in J^1. \tag{1.16}$$

Via vectors (1.16) we find

$$\widetilde{\varkappa}^0(t^1), \quad \widetilde{x}_j^{1,\alpha}, \quad j \in J^1 \quad (\varkappa^0(t^1) = \overline{x}^0(t^1)). \tag{1.17}$$

For sufficiently small $\theta > 0$, the radius $\overline{\rho}(t^1)$ of the ball into which the points (1.17) can be placed will be less than the analogous radius $\rho(t^1)$ for $\overline{x}^0(t^1)$, $x_j^{1,\alpha}$, $j \in J^1$. Having chosen an accuracy $\varepsilon > 0$ of the solution of problem (1.1), by continuing the corrections of normals of active constraints, one can obtain $\overline{\rho}(t^1) \leq \varepsilon$. A similar process of correction can be applied in the remaining moments of closing $t^2, \dots, t^p$. The control $\overline{u}^0(t)$, $t \in T$, under which $\overline{\rho}(t^i) \leq \varepsilon$, $i = \overline{1,p}$, is taken for the optimal program control $u^0(t)$, $t \in T$, of problem (1.1) if $\gamma^* < \varepsilon$. Otherwise we assume that problem (1.1) fails to possess admissible controls.

The algorithm of the work of an optimal regulator which in the real-time mode constructs a realization $u^*(t)$, $t \in T$, of the optimal feedback (1.9), consists of the two parts: 1) the algorithm of the start stage ($\tau = t_*$); 2) the algorithm of the current stage $\tau > t_*$. On the start stage, using the program solution $u^0(t)$, $t \in T$, the regulator sets $u^*(t_*) = u^0(t_*)$. The program solution can be obtained before the start of the control process and therefore we can impose no restrictions upon the time of its computing.

Let $\tau \in T_h$, $t^i \le \tau < t^{i+1}$, be the current moment. If at this moment the state $x(\tau)$ is not available, then we put $u^*(\tau) = u^0(\tau \mid \underline{\tau}(\tau), x(\underline{\tau}))$, i. e., without computation we use the value of the optimal program control already computed at the moment $\underline{\tau}(\tau)$ for the state $x(\underline{\tau})$. Let us consider the case $\underline{\tau}(\tau) = \tau$ where at the moment τ the state $x(\tau)$ is available. This state is the result of the state $x(\underline{\tau}(\tau))$ after the actions upon the system of both the control $u^*(t)$, $t \in [\underline{\tau}, \tau[$, and the perturbation $w(t)$, $t \in [\underline{\tau}, \tau[$. For small $(\tau - \underline{\tau})$, the state $x(\tau)$ is rather close to the state $x_0(\tau)$, to which the deterministic system (1.4) passes from $x(\underline{\tau})$ under the action of the control $u^*(t)$, $t \in [\underline{\tau}, \tau[$. In this case, the method in [5] rapidly constructs $u^0(t \mid \tau, x(\tau))$, $t \in [\tau, t^*]$, correcting $u^0(t \mid \underline{\tau}, x(\underline{\tau}))$, $t \in [\tau, t^*]$. If the vectors $x(\tau)$, $x_0(\tau)$ are essentially distinct from each other, then the time of correction can be decreased by means of parallel computations. Using the above algorithm and modern microprocessors, the optimal regulator can realize the feedback for dynamical systems of sufficiently high order.

Remark 1.2. For $T^p = T_h$, the optimal closable feedback (1.9) is called the optimal closable ("true") feedback. The classical method for construction of the optimal closed feedback is the dynamical programming (see [6]). The approach described above allows us to construct also the optimal closed feedbacks. In addition, the requirements upon the Random Access Memory are essentially lesser that those in the dynamical programming in view of the use of correction procedures in the control process. Exaggerated requirement to the volume of the memory in the dynamical programming can be explained by the following reasons: In its realization an additional work in the process of control is not foreseen, all information about the position solution is prepared beforehand for all thinkable positions.

2. Synthesis of optimal controls of the type of direct coupling

In the control over complex systems which function in the environment of permanently acting perturbations, the results of direct and indirect measurements of some perturbations often turn to be available. The utilization of that information in the formation of controlling actions can significantly augment the efficiency of a controlling system. Controls constructed by perturbations (input signals) are called controls of direct coupling type or compensational controls and differ from feedback controls which are constructed by the output signals.

Let us replace $w = w(t)$ in problem (1.1) with $v = v(t)$, $t \in T$; next, in solving problem (1.1) we will assume: 1) before the start of a control process it is known that in the control process in the capacity of a perturbation any discrete function $v(t) \in V$, $t \in T$, $V = \{v \in R : |v| \le 1\}$, can be realized; 2) in the control process at every current moment of time $\tau \in T_h$ the current value $v = v(\tau) \in V$ of perturbation will be available. Let us immerse problem (1.1) into the family of problems

$$c'x(t) \to \max, \quad \dot{x} = A(t)x + b(t)u + d(t)v, \quad x(\tau) = z, \quad x(t^*) \in X^*,$$
$$u(t) \in U \ \ , v(t) \in V, \quad t \in T(\tau) = [\tau, t^*],$$

which depends on the collection

$$\{\tau, z, v\}, \tag{2.1}$$

where $\tau \in T_h$ is the current moment of time, $z = x(\tau) \in R^n$ the current state, $v = v(\tau) \in V$ the value of the perturbation at the moment τ. The vector z in (2.1) is assumed to be known at the moment τ, since it can be constructed via the control $u^*(t)$, $t \in [t_*, \tau[$, already used before and the measured perturbation $v^*(t)$, $t \in [t_*, z[$:

$$z = F(\tau, t_*)x_0 + \int_{t^*}^{\tau} F(\tau, s)b(s)u^*(s)ds + \int_{t^*}^{\tau} F(\tau, s)d(s)v^*(s)ds$$
$$(F(t,s) = F(t)F^{-1}(s), \quad \dot{F} = A(t)F, \quad F(0) = E).$$

An available control $u_\tau(\cdot \mid \tau, z, v) = (u(t \mid \tau, z, v) \in U,\ t \in T(\tau))$ is called an admissible program control for the position (τ, z, v) if all generated by it, by the perturbation $v(\tau) = v$, and by perturbations $v_{\tau+h}(\cdot) = (v(t) \in V,\ t \in T(\tau + h))$ trajectories $x(t \mid \tau, z, u_\tau(\cdot \mid \tau, z, v), v, v_{\tau+h}(\cdot))$, $t \in T(\tau)$, at moment t^* hit the terminal set X^*. Denote

$$X_{t^*}(u_\tau(\cdot \mid \tau, z, v)) = \{x : x = x(t^* \mid \tau, z, v, u_\tau(\cdot \mid \tau, z, v), v_{\tau+h}(\cdot)),\ v(t) \in V,\ t \in T(\tau + h)\}.$$

We estimate the quality of the admissible control $u_\tau(\cdot \mid \tau, z, v)$ by the number

$$J(u) = \min c'x, x \in X_{t^*}(u_\tau(\cdot \mid \tau, z, v)). \tag{2.2}$$

An admissible control $u^0_\tau(\cdot \mid \tau, z, v)$ is considered as an optimal program control for the position (τ, z, v) if this control supplies the maximum to the value (2.2):

$$J(u^0) = \max_u J(u).$$

Let X_τ be the set of all $z \in R^n$ for which there exists $u^0_\tau(\cdot \mid \tau, z, v)$ for a fixed $\tau \in T_h$ and for all $v \in V$.

Definition 2.1. By an optimal control of the type of breakable direct discrete coupling we will call the function

$$u^0(\tau, z, v) = u^0(\tau \mid \tau, z, v), \quad z \in X_\tau, \quad v \in V, \quad \tau \in T_h. \tag{2.3}$$

The trajectory of the closed system

$$\dot{x} = A(t)x + b(t)u^0(t, x, v) + d(t)v, \quad x(t_*) = x_0,$$

is defined as a solution of the linear equation

$$\dot{x} = A(t)x + b(t)u^*(t) + d(t)v, \quad x(t_*) = x_0,$$

with the control

$$u^*(t) = u^0(t_* + kh, x(t_* + kh), v(t_* + kh)), \tag{2.4}$$
$$t \in [t_* + kh,\ t_* + (k+1)h[, \quad k = \overline{0, N-1}.$$

As in the preceding Section, we call function (2.4) the *realization* of the direct coupling (2.3). We will say that the direct coupling is realized in the real-time mode if at every current moment $\tau \in T_h$ the time for computing the value $u^*(\tau)$ by the known z, v does not exceed h. The device which is able to execute this work is called an optimal regulator.

We define the sets

$$X^1_\tau = \Big\{x \in R^n : x = \int_{\tau+h}^{\tau} F(t^*, s)d(s)v(s)ds, v(t) \in V,\ t \in T(\tau + h)\Big\},$$
$$X^2_\tau = \{x \in R^n : x + X^1_\tau \subset X^*\},$$
$$X^{v,z}_\tau = \Big\{x \in R^n : x = F(t^*, \tau)z + v\int_{\tau}^{\tau+h} F(t^*, s)d(s)ds,\ v \in V\Big\},$$
$$X^3_\tau = \Big\{x \in R^n : x = \int_{\tau}^{t^*} F(t^*, s)b(s)u(s)ds, u(t) \in U,\ t \in T(\tau)\Big\}.$$

In terms of these sets the condition of the admissibility of a control $u_\tau(\cdot \mid \tau, z, v)$ takes the form $X_\tau^{v,z} \subset (X_\tau^2 - X_\tau^3)$. In a similar way we define the set $X_\tau^{2,\alpha}$ by replacing X^* with $X^{*,\alpha} = X^* \cap \{x \in R^n : c'x \geq \alpha\}$. The maximal $\alpha = \alpha^0$, under which the inclusion is fulfilled

$$X_\tau^{v,z} \subset (X_\tau^{2,\alpha} - X_\tau^3), \tag{2.5}$$

is equal to the optimal guaranteed quality criterion of the problem. The control with which relation (2.5) is fulfilled for $\alpha = \alpha^0$, is equal to $u^0(t \mid \tau, z, v)$, $t \in T(\tau)$.

Remark 2.1. One can easily note that the definitions and the method of solving will not change principally if we replace assumption 2) with the requirement that, in the process of control, at every current moment $\tau \in T_h$ by the results of measurement up to the moment τ inclusively the forecast $\widehat{v}(t)$, $t \in [\tau, \tau + \theta[$, $\theta = lh$, $l = \overline{1, l^*}$, of the perturbation $v(t)$, $t \in [\tau, \tau + \theta]$, is available with the error $\overline{v}(t)$, $t \in [\tau, \tau + \theta] : v(t) = \widehat{v}(t) + \widetilde{v}(t)$, $|\widehat{v}(t)| \leq \overline{v}(t)$, $t \in [\tau, \tau + \theta]$.

For a constructive development of an optimal program control on the basis of the optimal control of the direct coupling type, we use the polyhedral approximation of relation (2.5), solve the approximation problem, and realize the finishing of the solution obtained up to the solution of the initial problem (1.1) (see Section 1 and assumptions in Section 2). As has been noted above, the realization of optimal controls of the type of breakable coupling is based on rapid corrections of program solutions. The information about some elements of program solution, which is obtained in the position $(\tau, x^*(\tau), v^*(\tau))$, is transformed by the dual method (see [5]) into the information for the position $(\tau + h, x^*(\tau + h), v^*(\tau + h))$, when the process of control passed from the moment τ into the next moment $\tau + h$ and $x^*(\tau + h)$, $v^*(\tau + h)$ have become known. Modern microprocessors may execute the work of correction within the time lesser than h even for control systems of rather high order. This makes it possible to construct on their basis the algorithm of functioning of the optimal regulator.

Remark 2.2. If we complete the a priori information with respect to the problem under consideration and assume that prior to the start of process of control at the moments of closing $T^p = \{t^i \in T_h,\ i = \overline{1, p}\}$, $t_* < t^1 < \cdots < t^p < t^*$, we know the values $v(t^i)$, $i = \overline{1, p}$, of perturbation, then we can introduce the notion of an optimal closable direct discrete coupling. The use of controls of that kind extends the set of initial states x_0, for which the solution of the problem of guaranteed optimization exists. The algorithm of the work of the optimal regulator which realizes the optimal control of the type of closable direct coupling can be constructed along the scheme of the algorithm from Section 1.

3. Synthesis of optimal controls of the type of combined coupling

3.1. *Method of solving.* In the control over dynamical systems functioning in the environment of permanently acting perturbations, in addition to the principle of the feedback and the principle of direct coupling, the principle of combined coupling (principle of direct-feedback coupling) is also used. As we have noted above, the combined principle of control is based on both those first principles and is realized as feedback and direct coupling which uses both the output and input signals. In what follows the results obtained in Sections 1 and 2 are extended to the case of an optimal combined control.

Assume that in the class of discrete controls $u(t)$, $t \in T$, perturbations $v(t)$, $t \in T$, and piecewise continuous perturbations $w(t)$, $t \in T$, the following problem is considered:

$$\begin{gathered} c'x(t^*) \to \max, \quad \dot{x} = A(t)x + b(t)u + d(t)v + f(t)w, \quad x(t_*) = x_0, \\ x(t^*) \in X^* = \{x \in R^n : g_* \leq Hx \leq g^*\}, \quad u(t) \in U = \{u \in R : |u| \leq 1\}, \\ v(t) \in V = \{v \in R : |v| \leq 1\}, \quad w(t) \in W = \{w \in R : |w| \leq 1,\ t \in T\} \end{gathered} \tag{3.1}$$

($x \in R^n$, $H \in R^{m \times n}$, $A(t)$, $b(t)$, $d(t)$, $f(t)$, $t \in T$, are piecewise continuous functions).

Suppose that before the start of the process of control we know that 1) in the process of control any discrete perturbations $v(t) \in V$, $t \in T$, and piecewise perturbations $w(t) \in W$, $t \in T$, can be realized; 2) at every current moment of time $\tau \in T_h$ we will know the current state $x(\tau)$ and the value $v(\tau)$ of the perturbation $v(t)$, $t \in T$. Let us immerse problem (3.1) into the family of analogous problems

$$c'x(t^*) \to \max, \quad \dot{x} = A(t)x + b(t)u + d(t)v + f(t)w, \quad x(\tau) = z, \quad v(\tau) = v, \tag{3.2}$$
$$x(t^*) \in X^*, \quad u(t) \in V, \quad v(t) \in U, \quad w(t) \in U, \quad t \in T(\tau + h),$$

which depends on the triple $\tau \in T_h$, $z \in R^n$, $v \in V$.

By an admissible control $u(\cdot \mid \tau, z, v) = (u(t \mid \tau, z, v) \in U,\ t \in T(\tau))$ we will call the program control for the position (τ, z, v) if the trajectories $x(t \mid \tau, z, v,\ u(\cdot \mid \tau, z, v),\ v_{\tau+h}(\cdot),\ w_\tau(\cdot))$, $t \in T(\tau)$, of system (3.2), which are generated by this control and all possible perturbations $v_{\tau+h}(\cdot) = (v(t),\ t \in T(\tau+h))$, $w_\tau(\cdot) = (w(t), t \in T(\tau))$, at the moment t^* hit the terminal set X^*. We denote $X_{t^*}(u(\cdot \mid \tau, z, v)) = \{x \in R^n : x = x(t^* \mid \tau, z, v,\ u(\cdot \mid \tau, z, v),\ v_{\tau+h}(\cdot),\ w_\tau(\cdot)), v(t) \in V, t \in T(\tau+h);$ $w(t) \in W, t \in T(\tau)\}$. We evaluate the quality of an admissible control by the number

$$J(u(\cdot \mid \tau, z, v)) = \min c'x, \quad x \in X_{t^*}(u(\cdot \mid \tau, z, v)). \tag{3.3}$$

An admissible control $u^0(t \mid \tau, z, v)$, $t \in T(\tau)$, will be called the optimal program control of problem (3.1) if among all possible controls it supplies the most meaning to the value (3.3): $J(u^0(\cdot \mid \tau, z, v)) = \max\limits_u J(u(\cdot \mid \tau, z, v))$.

Let X_τ be the set of vectors $z \in R^n$, for which problem (3.2) has a program solution $u^0(t \mid \tau, z, v)$, $t \in T(\tau)$, for every fixed $\tau \in T_h$ and all $v \in V$.

Definition 3.1. The function

$$u^0(\tau, z, v) = u^0(\tau \mid \tau, z, v), \quad z \in X_\tau, \quad v \in V, \quad \tau \in T_h, \tag{3.4}$$

is called the optimal control of the type of direct-feedback (combined) discrete coupling for problem (3.1).

We define the trajectory of the closed nonlinear system $\dot{x} = A(t)x + b(t)u^0(t, x, v) + d(t)v + f(t)w$, $x(t_*) = x_0$, which results from (3.1) after replacement of the control u with function (3.4), as a solution of the linear equation $\dot{x} = A(t)x + b(t)u^*(t) + d(t)v + f(t)w$, $x(t_*) = x_0$, with the control

$$u^*(t) = u^0(t_* + kh,\ x(t_* + kh),\ v(t_* + kh)),$$
$$t \in [t_* + kh, t_* + (k+1)h[, \quad k = \overline{0, N-1}.$$

As above, we introduce the notion of the *realization* of direct-feedback coupling (3.4) in a concrete process of control.

We introduce the sets

$$X_\tau^1 = \Big\{x \in R^n : x = \int_{\tau-h}^{t^*} F(t^*, s)d(s)v(s)ds + \int_\tau^{t^*} F(t^*, s)f(s)w(s)ds,$$
$$v(t) \in V, \quad t \in T(\tau + h), \quad w(t) \in W, \quad t \in T(\tau)\Big\},$$
$$X_\tau^2 = \{x \in R^n : x + X_\tau^1 \subset X^*\},$$
$$X_\tau^{v,z} = \Big\{x \in R^n : x = F(t^*, \tau)z + v\int_\tau^{\tau+h} F(t^*, s)d(s)ds, \ v \in V\Big\}, \tag{3.5}$$
$$X_\tau^3 = \Big\{x \in R^n : x = \int_\tau^{t^*} F(t^*, s)b(s)u(s)ds, \ u(t) \in U, \ t \in T(\tau)\Big\}.$$

In terms of sets (3.5) the condition for admissibility of the program control takes the form $X_\tau^{v,z} \subset (X_\tau^2 - X_\tau^3)$. A geometrical interpretation of the optimality condition of an admissible control can be obtained as follows. Let us introduce the set $X^{*,\alpha} = X^* \cap \{x \in R^n : c'x \geq \alpha\}$. Put instead of X^2_τ: $X^{2,\alpha}_\tau = \{x \in R^n : x + X^1_\tau \subset X^{*,\alpha}\}$. The admissible program control provides the quality criterion of problem (3.1) with the guaranteed value α if

$$X^{v,z}_\tau \subset (X^{2,\alpha}_\tau - X^3_\tau). \tag{3.6}$$

The most $\alpha = \alpha^0$ with which inclusion (3.6) holds is equal to the optimal (most) guaranteed value of the quality criterion. The admissible control, on which condition (3.6) is fulfilled with $\alpha = \alpha_0$, is the optimal program control for the position (τ, z, v).

The constructive verification of relation (3.6) and the development of the optimal program control consist of the two procedures: 1) solving auxiliary linear optimal control problems constructed on the basis of polyhedral approximations of the sets in (3.5) and replacement of relation (3.6) with the approximate one

$$\max_{x\in X^{v,z}_\tau} f'_j x \le \max_{x\in X^{2,\alpha}_\tau,\ y\in X^3_\tau} f'_j(x-y), \quad j=\overline{1,q},$$

where f_j, $j = \overline{1,q}$, is a collection of unit vectors (see Section 1); 2) the correction (finishing) which represents a process of consecutive refinement of solution of auxiliary problems up to the construction of the solution of problem (3.1) with a prescribed accuracy.

The algorithm of work of the optimal regulator represents the procedure of correction at the current position $(\tau, x^*(\tau), v^*(\tau))$ of elements of the program solution constructed for the preceding position $(\tau - h, x^*(\tau - h), v^*(\tau - h))$. In the construction of optimal program controls which realize the optimal control of the type of combined coupling (3.4), the defining role is played by the rapid algorithms (see [5]).

3.2. *Closed combined coupling and the method of dynamical programming.*

Definition 3.2. For $T^p = T_h$, the optimal control of the type of closable combined coupling will be called the optimal control of the type of closed combined coupling.

Closed couplings use the greatest volume of a priori information and therefore are considered as the most effective. Since we nowhere above used the condition $T^p \ne T_h$ in the exposition of the method for synthesis of optimal closable combined couplings, the synthesis of optimal controls of the type of closed combined coupling is to be carried out by the scheme described above.

The classical method for synthesis of optimal closed coupling is the dynamical programming (see [6]). For the problem under investigation this method leads us to the Bellman equation

$$B_\tau(z) = \min_{v\in V}\ \max_{u\in U}\ \min_{w(s)\in W,\ s\in[\tau,\tau+h]} B_{\tau+h}\Big(F(\tau+h,\tau)z + u\int_\tau^{\tau+h} F(\tau+h,s)b(s)ds + \\ + v\int_\tau^{\tau+h} F(\tau+h,s)d(s)ds + \int_\tau^{\tau+h} F(\tau+h,s)f(s)w(s)ds\Big) \tag{3.7}$$

with the initial condition

$$B_{t^*}(z) = \begin{cases} c'z, & z \in X^*; \\ -\infty, & z \notin X^*, \end{cases}$$

and to the relation for calculation of the optimal control of the type of closed combined coupling $u^0(\tau, z, v)$, $z \in X_\tau$, $v \in V$, $\tau \in T_h$,

$$\min_{w(s)\in W,\ s\in[\tau,\tau+h]} B_{\tau+h}\Big(F(\tau+h,\tau)z + u^0(\tau,z,v)\int_\tau^{\tau+h} F(\tau+h,s)b(s)ds + \\ + v\int_\tau^{\tau+h} F(\tau+h,s)d(s)ds + \int_\tau^{\tau+h} F(\tau+h,s)f(s)w(s)ds\Big) = \\ = \max_{u\in U}\ \min_{w(s)\in W,\ s\in[\tau,\tau+h]} B_{\tau+h}\Big(F(\tau+h,\tau)z + u\int_\tau^{\tau+h} F(\tau+h,s)b(s)ds + \\ + v\int_\tau^{\tau+h} F(\tau+h,s)d(s)ds + \int_\tau^{\tau+h} F(\tau+h,s)f(s)w(s)ds\Big). \tag{3.8}$$

Here $F(t,s) = F(t)F^{-1}(s)$, $\dot{F} = A(t)F$, $F(0) = E$, $X_\tau = \{z \in R^n : B_\tau(z) \neq -\infty\}$. As is seen in (3.7), (3.8), the Bellman equation can be solved step-by-step from the left to the right by tabulating in addition the Bellman operator $B_1(z)$, $z \in X_\tau$, $\tau \in T_h$, and the optimal control $u^0(\tau, z, v)$, $z \in X_\tau$, $v \in V$, $\tau \in T_h$. This task can be realized before the start of the process of control, since in (3.7), (3.8) only a priori information is used. However, as is well-known from [6], it is very difficult to realize the method for construction of highly accurate solutions of the problem with $n \geq 3$ in view of the exaggerated volume of the required RAM.

The principal difference between the method exposed in this article and that of dynamical programming is the following. On the preliminary stage (before the start of the process of control) a rough information concerning the solution should be prepared with the use of rather modest volume of memory; afterwards, within the process of control, this information at every current moment of time $\tau \in T_h$ is rapidly corrected (see [5]), which depends on both the realized state $x^*(\tau)$ and perturbation $v^*(\tau)$. In [5], [7] a series of rapid algorithms were developed, which make it possible to realize the correction in the real-time mode on modern computing devices for systems of control of rather high order.

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