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ESTIMATES OF UPPER AND LOWER LOGARITHMIC DENSITIES IN THE THEORY OF MEROMORPHIC FUNCTIONS

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In this article we use the standard notation of the theory of distribution of values $m(r,a,f)$, $N(r,a,f)$, $T(r,f)$, $\delta(a,f)$ (see [1], [2]). R. Nevanlinna obtained the two basic theorems of this theory.

Theorem A. *Let $f(z)$ be a meromorphic function on the plane $\mathbb{C}$. Then, for every $a \in \overline{\mathbb{C}}$,*

$$m(r,a,f) + N(r,a,f) = T(r,f) + O(1), \quad r \to \infty. \tag{1}$$

Theorem B. *For a transcendent meromorphic function $f(z)$ and any finite set of distinct complex numbers $\{a_k\}_{k=1}^q \in \overline{\mathbb{C}}$, the inequality*

$$\sum_{k=1}^{q} m(r,a_k,f) \le (2+o(1))T(r,f), \quad r \to \infty, \tag{2}$$

is fulfilled for all r with possible except for a set of finite measure.

From relations (1), (2) it follows that $\delta(a,f) \le 1$ and $\sum\limits_{a\in\overline{\mathbb{C}}} \delta(a,f) \le 2$.

In 1969 V.P. Petrenko stated the question: How will change the Nevanlinna theory if one will consider an approximation of the meromorphic function $f(z)$ to the value a in another metric? In this connection he introduced the following function:

$$\mathcal{L}(r,a,f) = \max_{|z|=r} \log^+ \frac{1}{|f(z)-a|}, \quad \mathcal{L}(r,\infty,f) = \max_{|z|=r} \log^+ |f(z)|.$$

The value $\beta(a,f) = \liminf\limits_{r\to\infty} \frac{\mathcal{L}(r,a,f)}{T(r,f)}$ is called the value of the deviation of the meromorphic function $f(z)$ at the point a.

The function $\mathcal{L}(r,a,f)$ characterizes the approximation of the function f to the value a on the circle $\{z : |z| = r\}$ in the uniform metric, and $m(r,a,f)$ — in the metric of $L_1[0,2\pi]$. Thus, the value $\beta(a,f)$ characterizes the approximation of the function $f(z)$ to the value a in the metric stronger than $\delta(a,f)$. Nevertheless, it turned out that for meromorphic functions of finite lower order $\lambda = \liminf\limits_{r\to\infty} \frac{\log T(r,f)}{\log r}$ the properties of the values $\beta(a,f)$ resemble those of $\delta(a,f)$. Thus, in [3] the exact above estimate for $\beta(a,f)$ was obtained, and also a certain estimate for $\sum\limits_{(a)} \beta(a,f)$. We denote by $\Phi(\lambda)$ the class of meromorphic functions of finite lower order λ, and by $\Phi(\lambda,\rho)$ the class of meromorphic functions of finite lower order λ and order $\rho = \limsup\limits_{r\to\infty} \frac{\log T(r,f)}{\log r}$.

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