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THE UNITS OF CYCLIC GROUPS OF ORDERS 7 AND 9

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1. Introduction

The present article was written basing on the report made by the authors on International Conference "Algebra and Analysis", dedicated to N.G. Chebotaryov's centenary (Kazan, June, 1994; see [1]). We shall consider here integral group rings, for the sake of brevity their units (= invertible elements) will be called units of the respective groups. Complete information concerning the groups of units of cyclic groups of orders 5, 8, 10, and 12 can be found in [2]. In the present article we describe the groups of units of cyclic groups of orders 7 and 9. Thus, this results in a description of groups of units of all the cyclic groups of orders not exceeding 10, because for other orders the units are trivial.

2. Units of a cyclic group of order 7

Let α be the primitive root of order 7 of the unity, $\eta_1 = \alpha + \alpha^{-1}$, and $\eta_2 = \alpha^2 + \alpha^{-2}$. Let $\mathbb{Z}[\alpha]$ be a ring of cyclotomic integers of order 7, and $U(\mathbb{Z}[\alpha])$ its group of units.

Lemma 1 (see [3], p. 241). $U(\mathbb{Z}[\alpha]) = \langle -1\rangle \times \langle\alpha\rangle \times \langle\eta_1\rangle \times \langle\eta_2\rangle$.

Let $\mathbb{Z}\langle x\rangle$ be an integral group ring of the cyclic group $\langle x\rangle$ of order 7, and $U(\mathbb{Z}\langle x\rangle)$ its group of units. Let

$$\varphi : \mathbb{Z}\langle x\rangle \to \mathbb{Z}[\alpha] \quad \varphi\Big(\sum_{i=0}^{6} a_i x^i\Big) = \sum_{i=0}^{6} a_i \alpha^i,$$
$$\psi : \mathbb{Z}\langle x\rangle \to \mathbb{Z} \quad \psi\Big(\sum_{i=0}^{6} a_i x^i\Big) = \sum_{i=0}^{6} a_i.$$

These mappings are homomorphisms of rings.

Lemma 2. *If u_1 is a unit of the ring $\mathbb{Z}\langle x\rangle$ and $\varphi(u_1) = \eta_1^k$, then k is divisible by 3. Moreover, if $\varphi(u_1) = \eta_1^3$, then $u_1 = -1 + 2x - x^2 - x^5 + 2x^6$ and $u_1^{-1} = -3 + x + 3x^2 - 2x^3 - 2x^4 + 3x^5 + x^6$.*

Proof. Let $\varphi(u_1) = \eta_1$. Then $u_1 \in x + x^6 + \ker\varphi$. Since $\ker\varphi = \Big\{\sum_{i=0}^{6} a_i x^i \big| a_i = a_6$ for all $i = 1,\dots,5\Big\}$, we have $u_1 = x + x^6 + a\sum_{i=0}^{6} x^i$. But then $\psi(u_1) = 2 + 7a = \pm 1$, which is impossible. Let $\varphi(u_1) = \eta_1^3 = \alpha^3 + 3\alpha + 3\alpha^{-1} + \alpha^{-3}$. Then $u_1 = 3x + x^3 + x^4 + 3x^6 + a\sum_{i=0}^{6} x^i$ and $\psi(u_1) = 8 + 7a = \pm 1$. Hence $a = -1$ and $u_1 = 3x + x^3 + x^4 + 3x^6 - \sum_{i=0}^{6} x^i = -1 + 2x - x^2 - x^5 + 2x^6$. □