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Almost Hermitian Structures on the Tangent Bundle of Almost Symplectic Manifold

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In this paper we find invariant characteristics for the Gray–Hervella classes of almost Hermitian structures which naturally arise on the total space of the tangent bundle of an almost symplectic manifold.

1. Let M be a smooth n-dimensional manifold, x^i be local coordinates on M, and $\omega = \frac{1}{2}\omega_{ij}dx^i \wedge dx^j$ be a nondegenerate differential 2-form, i.e., an almost symplectic structure on M $(i, j, k, \dots = 1 \dots n;$ $\omega_{ij} = -\omega_{ji}$, $\det \|\omega_{ij}\| \neq 0)$. A linear connection ∇ is said to be compatible with ω if $\nabla_X \omega = 0$ for any vector field X on M, or, with respect to the local coordinates,

$$\partial_i \omega_{jk} - \omega_{pk}\Gamma^p_{ij} - \omega_{jp}\Gamma^p_{ik} = 0, \tag{1}$$

where Γ^k_{ij} are the connection coefficients of ∇. Taking the cyclic permutation of the indices in (1) and summing up the resulting equations, we obtain

$$\omega_{pk}S^p_{ij} + \omega_{pj}S^p_{ki} + \omega_{pi}S^p_{jk} = \partial_i\omega_{jk} + \partial_j\omega_{ki} + \partial_k\omega_{ij},$$

where $S^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}$ are the components of the torsion tensor S of the connection ∇. Hence follows that, if an almost symplectic structure is not a symplectic structure ($d\omega \neq 0$), then the connection necessarily has torsion.

For a given symplectic structure ω, infinitely many connections compatible with ω exist. The coefficients of these connections can be written as follows [1]:

$$\Gamma^k_{ij} = \frac{1}{2}\omega^{kp}(\partial_i\omega_{pj} + \eta_{pji}),$$

where η_{pji} is a system of functions such that $\eta_{pji} = \eta_{jpi}$. This system determines a differential geometric object whose transformation law provides the appropriate transformation law for Γ^k_{ij}.

2. Let TM be the tangent bundle of an almost symplectic manifold (M, ω), and (x^i, y^i) be the natural local coordinates on TM. The vector fields $\delta_I = (\delta_i, \dot{\partial}_i)$, $\delta_i = \partial_i - N^k_i\dot{\partial}_k$, $N^k_i = \Gamma^k_{ip}y^p$, $\partial_i = \partial/\partial x^i$, $\dot{\partial}_i = \partial/\partial y^i$ form a local frame field on TM adapted to the almost product structure $T_z(TM) = H_z \oplus V_z$, $z \in TM$.

The structure equations are written as follows: $[\delta_I, \delta_J] = R^K_{IJ}\delta_K$, $I, J, K, \dots = 1, \dots, 2n$, where R^K_{IJ} are the components of the nonholonomity object: $R^{n+k}_{ij} = -y^m R^k_{ijm}$, $R^{n+k}_{in+j} = \Gamma^k_{ij}$, $R^{n+k}_{n+ij} = -\Gamma^k_{ij}$, $R^k_{ij} = R^k_{in+j} = R^k_{n+ij} = R^k_{n+in+j} = R^{n+k}_{n+in+j} = 0$. Here R^k_{ijm} are the coordinates of the curvature tensor of ∇.

The horizontal distribution on TM determined by the connection ∇ determines, in its turn, the canonical almost complex structure on TM: $J : J^2 = -\mathrm{id}$, $JX^h = X^v$, $JX^v = -X^h$, where X^h and X^v are respectively the horizontal and vertical lifts of a vector field X on the base M.

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