Russian Mathematics (Iz. VUZ) *Izvestiya VUZ. Matematika*
Vol. 44, No. 6, pp.50–59, 2000 *UDC 512.573*

MATRIX LINEAR OPERADS

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In this article we shall start a detailed study of a family of linear operads and varieties of algebras over them. Many known classical objects, e. g., multidimensional matrices, tensors, etc., admit an interpretation in terms of the operad theory. The latter can be treated as a multidimensional analog of the theory of associative rings and modules. Seemingly, as a subject of studying, the linear operads first appeared (under a different name) in [1]. The proper term "operad" appeared in [2] (see also [3]). As concerns the present state of both the operad theory and its applications, one can refer to [4]–[8]. In [9]–[12], it was shown that the class of varieties of linear (multioperator) algebras over linear operads exactly coincides with the class of varieties of multioperator linear algebras determined by multilinear identities.

We shall begin this article with basic definitions concerning the theory of both operads and algebras over them. We shall describe an operad analog for group rings and algebras over operads of that kind. We define the main class of objects under investigation: The operads of multidimensional matrices and their generalizations. We establish multidimensional analogs for the properties of the matrix rings and construct a one-to-one correspondence between the congruences of operads and those of the matrix operads, which generalizes the corresponding fact of the theory of associative rings. Finally, we shall prove that the two categories of algebras over 1) a given operad and 2) the corresponding matrix operad are equivalent. It will be proved that, along with the functors which realize the equivalence between the categories (varieties) of the algebras over an operad and those over the corresponding matrix operad, the functors realizing the equivalence between the categories of modules over the corresponding algebras exist. The results of this article were announced in part in [13] and [14].

To designate a function we shall use both the ways: to the left and to the right of the proper argument. In particular, we shall use both the left and right actions of the permutation group Σ_n on the set $[n]=\{1,\dots,n\}$, and $\sigma i = i\sigma^{-1}$, where $\sigma\in\Sigma_n$, $i\in[n]$. For a given action of Σ_n on a set X, in a similar way we define the corresponding action of Σ_n on X in the opposite side. Hence we can define the action of Σ_n on the sets of the form X^n and on the sets of functions of n variables as follows. We set $\sigma(x_1,x_2,\dots,x_n)=(x_{1\sigma},x_{2\sigma},\dots,x_{n\sigma})$, $(x_1,x_2,\dots,x_n)\sigma=(x_{\sigma1},x_{\sigma2},\dots,x_{\sigma n})$. For example, given the mapping $f: X^n\to Y$, we define the mapping $f\sigma$ by $f\sigma(\overline{x})=f(\sigma\overline{x})$, $\overline{x}\in X^n$ (in the case where the function symbol f is written to the left of the argument $\overline{x}$), or the mapping σf by $(\overline{x})\sigma f=(\overline{x}\sigma)f$ (in case f is written to the right of $\overline{x}$). By the partitioning of the set $[n]$ into m parts we shall mean an ordered succession of positive integers $\alpha=(n_1,\dots,n_m)$ such that $|\alpha|=n_1+\dots+n_m=n$. A set of all these partitions is denoted by $P(n,m)$. The permutation group Σ_m acts on $P(n,m)$ on the right: $\alpha\sigma=(n_{\sigma1},\dots,n_{\sigma n})$. For $\alpha\in P(n,m)$ and $\sigma\in\Sigma_m$, we define $\alpha*\sigma\in\Sigma_n$ as follows. Let X be a sufficiently large set, $\overline{x}_1\in X^{n_1},\dots,\overline{x}_m\in X^{n_m}$, $\overline{x}_1\dots\overline{x}_m\in X^{n_1+\dots+n_m}$, $\sigma\in\Sigma_m$, $\alpha=(n_1,\dots,n_m)$. Then the permutation $\alpha*\sigma$ is uniquely determined via its action on $X^{n_1+\dots+n_m}$: $(\overline{x}_1\dots\overline{x}_m)(\alpha*\sigma)=\overline{x}_{\sigma1}\dots\overline{x}_{\sigma m}$. For $\tau_1\in\Sigma_{n_1},\tau_2\in\Sigma_{n_2},\dots,\tau_m\in\Sigma_{n_m}$, the

Supported by Russian Foundation for Basic Research (grant no. 96-01-00717).