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# On Symmetric Spaces With Convergence in Measure on Reflexive Subspaces

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**Abstract**—A closed subspace $H$ of a symmetric space $X$ on $[0,1]$ is said to be strongly embedded in $X$ if in $H$ the convergence in $X$-norm is equivalent to the convergence in measure. We study symmetric spaces $X$ with the property that all their reflexive subspaces are strongly embedded in $X$. We prove that it is the case for all spaces, which satisfy an analogue of the classical Dunford–Pettis theorem on relatively weakly compact subsets in $L_1$. At the same time the converse assertion fails for a broad class of separable Marcinkiewicz spaces.

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**Introduction.** We say that a closed subspace $H$ of a symmetric space $X$ on $[0,1]$ is *strongly embedded* in $X$ if in $H$ the convergence in $X$-norm is equivalent to the convergence in measure. A standard example is the subspace $[r_k]$ generated by the Rademacher functions $r_k(t) = \operatorname{sgn} \sin(2^k \pi t)$, $k = 1, 2, \ldots$, in $L_p[0,1]$, $1 \le p < \infty$. Indeed, in view of classical Khintchine's inequality (see, e.g., [1], Chap. 5, theorem 8.4), the sequence $\{r_k\}_{k=1}^{\infty}$ is equivalent in $L_p[0,1]$ to the canonical basis in $l_2$ for all $0 < p < \infty$. So ([2], proposition 6.4.5), the subspace $[r_k]$ is strongly embedded in $L_p[0,1]$.

First, a close notion was introduced by Rudin for the study of Fourier analysis in $L_p$-spaces on the group $[0, 2\pi)$ [3]. In this connection, a strongly embedded subspace of $L_p[0,1]$, $1 \le p < \infty$, usually is called a $\Lambda(p)$-space ([4], Chap. III, definition 6). For the first time this notion was used basically in harmonic analysis (here we mention only the famous theorems of Bourgain [5] and of Bachelis–Ebenstein [6] devoted to solving the so-called "$\Lambda(p)$-problem"). Next the concept proved to be extremely useful also in studying the geometry of Banach spaces (see, e.g., [2], Chap. 6, and [7]).

The present paper is devoted to the study of symmetric spaces (in short, s. s.) $X$ on $[0,1]$ such that every reflexive subspace of $X$ is strongly embedded in $X$ (in this case we say that $X$ has the property $(\Lambda_R)$). In 2008, Lavergne [8] showed that every Orlicz space $L_M$ possesses this property whenever it is situated sufficiently "close" to the space $L_1$; more exactly, if the complementary (to $M$) function $M'$ satisfies the condition $\lim\limits_{t\to\infty} \frac{M'(ct)}{M'(t)} = \infty$ for some $c > 1$ (the corresponding family of Orlicz spaces is denoted usually by $(\nabla_3)$). The proof of this result in [8] is based on using Alexopoulos' extension (see [9], corollary 2.9) to the family $(\nabla_3)$ of the classical Dunford–Pettis theorem about equicontinuity of norms of relatively weakly compact subsets of $L_1$ (see, e.g., [2], theorem 5.2.9). On the other hand, in [10], s. s., for which the latter extension holds, were characterized as the spaces, in which any sequence converging both weakly and in measure to zero converges in norm (such spaces are called sometimes $(Wm)$-spaces). In this connection the following natural question arises: Whether the above Lavergne's result about topological properties of reflexive subspaces holds for all symmetric $(Wm)$-spaces? We give here the positive answer to this question in Theorem 2.

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