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Explicit Forms of the Schwarz Integral with Applications for Inverse Boundary-Value Problems

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Abstract—We obtain new explicit forms of the Schwarz integral in the unit circle. By means of superpositions of logarithms we establish criteria for the appearance of singular boundary points and circles in inverse boundary-value problems.

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1. The Schwarz integral for the disk $|\zeta| < 1$ is

$$f(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} u(\theta)\frac{e^{i\theta}+\zeta}{e^{i\theta}-\zeta}d\theta. \tag{1}$$

It is closely related to the Cauchy-type integral

$$f(\zeta)+f(0) = \frac{1}{2\pi i}\int_{|t|=1}\frac{2u(\theta(t))dt}{t-\zeta},\ \ t=e^{i\theta},\ \ 0\le\theta\le 2\pi. \tag{2}$$

The known explicit methods for evaluation of these integrals (see the book by G. N. Pykhteev [1]) enable us to obtain their explicit forms. By virtue of formula (2) (it is mentioned in [1], P. 35), any explicit representations of Cauchy-type integrals can be rewritten for the corresponding Schwarz integrals. In [1] G. N. Pykhteev describes several approaches to the explicit evaluation of the Schwarz integrals and gives numerous examples of explicit formulae.

The following theorem is a generalization of example 2 in [1] (P. 49).

Theorem 1. *The following criterion is valid for integral* (1):

$$\left\{f(\zeta)=\frac{A_0+A_1\zeta+\cdots+A_n\zeta^n}{B_0+B_1\zeta+\cdots+B_n\zeta^n},\right.$$
$$\left.\text{the fraction is irreducible (the numerator and denominator have no common zeros)},\right.$$
$$\left.B_0B_n\ne 0,\ B_0+B_1\zeta+\cdots+B_n\zeta^n\ne 0,\ |\zeta|\le 1\right\}\Leftrightarrow\left\{u(\theta)=\frac{\sum\limits_{k=0}^{n}(a_k\cos k\theta+b_k\sin k\theta)}{\sum\limits_{k=0}^{n}(c_k\cos k\theta+d_k\sin k\theta)},\right.$$
$$\left.\text{the fraction is irreducible},\ c_n+id_n\ne 0,\ \sum_{k=0}^{n}(c_k\cos k\theta+d_k\sin k\theta)\ne 0,\ 0\le\theta\le 2\pi\right\}.$$

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