Russian Mathematics (Iz. VUZ) *Izvestiya VUZ. Matematika*
Vol. 46, No. 4, pp. 43–48, 2002 *UDC 517.55*

BOUNDARY BEHAVIOR OF AN INTEGRAL OF LOGARITHMIC RESIDUE TYPE

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Let D be a bounded domain in $\mathbb{C}^n$ with a piecewise smooth boundary ∂D and $w=\psi(z)$ a holomorphic mapping from $\overline{D}$ to $\mathbb{C}^n$, possessing a finite number of zeros E_ψ on $\overline{D}$. Let $B(z,R)=\{\zeta : |\zeta - z| < R\}$ be the ball centered at the point z of the radius $R>0$, while $S(z,R)=\partial B(z,R)$. Suppose that a is a zero of the mapping ψ and $B(a,R)$ does not contain other zeros of ψ. Then a ball $B(0,r)$ can be found such that for almost all points $\zeta\in B(0,r)$ the mapping $w=\psi-\zeta$ has the same number of zeros in $B(a,R)$. This number is called the multiplicity of the zero a of the mapping ψ (see, e.g., [1], § 2) and is denoted by μ_a.

For a point $z\in E_\psi\cap\partial D$ we consider the ball $B(z,R)$ which does not contain other zeros of ψ, and put

$$\tau_\psi(z)=\lim_{r\to+0}\frac{\mathcal{L}^{2n-1}[S(0,r)\cap\psi(B(z,R)\cap D)]}{\mathcal{L}^{2n-1}[S(0,r)]},$$

where $\mathcal{L}^{2n-1}$ is the $(2n-1)$-dimensional Lebesgue measure. In other words, we consider not a solid angle of the tangent cone to the domain D at the point z, but the solid angle of the tangent cone of the image $\psi(B(z,R)\cap D)$ at the point 0. (See the definition of the tangent cone in [2], § 3.1.21.)

For $z\in E_\psi$ and a sufficiently small neighborhood V_z of the point z, the set $B_\psi(z,r)=\{\zeta\in V_z : |\psi(\zeta)|<r\}$ is relatively compact in V_z, while the set $S_\psi(z,r)=\{\zeta\in V_z : |\psi(\zeta)|=r\}$ is a smooth $(2n-1)$-dimensional cycle (for almost all sufficiently small $r>0$) by Sard's theorem.

Let us define the principal value v.p.^ψ of the integral of a certain measurable function φ at the point $z\in E_\psi$ by the neighborhood S of the point z of the surface ∂D as follows:

$$\text{v.p.}^\psi\int_S\varphi(\zeta)d\mathcal{L}^{2n-1}(\zeta)=\lim_{r\to+0}\int_{S\setminus B_\psi(z,r)}\varphi(\zeta)d\mathcal{L}^{2n-1}(\zeta).$$

The definition of the principal value by Cauchy v. p. differs from the usual one by the following fact: We reject not a ball neighborhood of the point z, but the "distorted" ball $B_\psi(z,r)$.

Let us introduce the kernel $U(\psi(\zeta))$ which is used in the formula for the multidimensional logarithmic residue (see, e.g., [1], § 3). This kernel is obtained by the substitution $w=\psi(z)$ from the Bochner–Martinelli kernel $U(w)$. Let us recall that

$$U(w)=\frac{(n-1)!}{(2\pi i)^n}\sum_{k=1}^n(-1)^{k-1}\frac{\overline{w}_k d\overline{w}[k]\wedge dw}{|w|^{2n}},$$

where $d\overline{w}[k]=d\overline{w}_1\wedge\dots\wedge d\overline{w}_{k-1}\wedge d\overline{w}_{k+1}\wedge\dots\wedge d\overline{w}_n$, and $dw=dw_1\wedge\dots\wedge dw_n$. The kernel $U(\psi(\zeta))$ is a closed differential form of type $(n,n-1)$ on $\overline{D}$ with singularities in the points $a\in E_\psi$.

Let us formulate the principal result.

Supported by the Russian Foundation for Basic Research, grant no. 99-01-00790.

Theorem. *If a function F satisfies on $\overline{D}$ the Hölder condition with exponent $\gamma > 0$ (i. e., $F \in \mathcal{C}^\gamma(\overline{D})$) and is holomorphic in D, then*

$$\text{v. p.} \int_{\partial D} F(\zeta)U(\psi(\zeta)) = \sum_{a \in E_\psi \cap D} \mu_a F(a) + \sum_{a \in E_\psi \cap \partial D} \tau_\psi(a)\mu_a F(a).$$

This is the formula for the multidimensional logarithmic residue with singularities on domain's boundary. If the zeros of the mapping ψ do not lie on the boundary, then it turns into the ordinary formula of logarithmic residue in [1] (§ 3). For the case of simple zeros $a \in \partial D$, it gives us the theorem from [3]. In addition, this Theorem generalizes theorem 20.7 in [4], where additional constraints were imposed on both the boundary ∂D and the mapping ψ.

The following proposition in known (see [2], theorem 3.2.5):

Let the mapping $\psi : \mathbb{R}^m \to \mathbb{R}^n$ be Lipschitz and $m \le n$. Then

$$\int_A g(\psi(x)) J_m\psi(x)\, d\mathcal{L}^m(x) = \int_{\mathbb{R}^n} g(y) N(\psi|A, y)\ d\mathcal{H}^m(y) \qquad (1)$$

if the set A is $\mathcal{L}^m$-measurable, $g : \mathbb{R}^n \to \overline{\mathbb{R}}$, and $N(\psi|A, y) < \infty$ for $\mathcal{H}^m$-almost all y.

Here $J_m\psi(x)$ is the m-dimensional Jacobian of the mapping ψ, $\mathcal{L}^m$ the m-dimensional Lebesgue measure, $\mathcal{H}^m$ the m-dimensional Hausdorff measure, $N(\psi \mid A, y)$ the function of multiplicity of the mapping ψ, i. e., the number of preimages $\psi^{-1}(y)$ lying in A.

First we will prove Theorem for the principal value v. p.^ψ.

Lemma 1. *In the conditions of Theorem the equality is fulfilled*

$$\text{v. p.}^\psi \int_{\partial D} F(\zeta)U(\psi(\zeta)) = \sum_{a \in E_\psi \cap D} \mu_a F(a) + \sum_{a \in E_\psi \cap \partial D} \tau_\psi(a)\mu_a F(a).$$

Proof. In the domain $D_r = D \setminus \bigcup\limits_{a \in E_\psi \cap \partial D} B_\psi(a, r)$ by the formula for the multidimensional logarithmic residue (see [1], § 3) we have

$$\int_{\partial D_r} F(\zeta)U(\psi(\zeta)) = \sum_{a \in E_\psi \cap D} \mu_a F(a),$$

and

$$\text{v. p.}^\psi \int_{\partial D} F(\zeta)U(\psi(\zeta)) = \lim_{r \to +0} \int\limits_{\partial D \setminus \bigcup\limits_{a \in E_\psi \cap \partial D} B_\psi(a,r)} F(\zeta)U(\psi(\zeta)).$$

Therefore

$$\int_{\partial D_r} F(\zeta)U(\psi(\zeta)) = \int\limits_{\partial D \setminus \cup_a B_\psi(a,r)} F(\zeta)U(\psi(\zeta)) - \sum_a \int\limits_{S_\psi(a,r) \cap D} F(\zeta)U(\psi(\zeta));$$

besides,

$$\int\limits_{S_\psi(a,r) \cap D} F(\zeta)U(\psi(\zeta)) = \int\limits_{S_\psi(a,r) \cap D} (F(\zeta) - F(a))U(\psi(\zeta)) + F(a) \int\limits_{S_\psi(a,r) \cap D} U(\psi(\zeta)). \qquad (2)$$

In what follows we will use Loyasevich's inequality (see [5], p. 73)

$$|\zeta - a| \le C|\psi(\zeta)|^\alpha \qquad (3)$$

for some positive numbers α and C and points ζ from a sufficiently small neighborhood of the point a. Let us show that the first of the integrals in formula (2) tends to zero as $r \to +0$. Using the Hölder condition for the function F, equality (1), and inequality (3), we obtain

$$\int\limits_{S_\psi(a,r)\cap D} |F(\zeta) - F(a)| \frac{|\psi_k|}{|\psi(\zeta)|^{2n}} |d\overline{\psi}[k] \wedge d\psi| \le C_1 \int\limits_{S_\psi(a,r)\cap D} |\psi(\zeta)|^{\gamma\alpha+1-2n} |d\overline{\psi}[k] \wedge d\psi| \le$$

$$\le C_1 \mu_a \int\limits_{S(0,r)\cap\psi(D)} |w|^{\gamma\alpha+1-2n} d\mathcal{H}^{2n-1}(w) \le C_2 \int\limits_{S(0,r)} |w|^{\gamma\alpha+1-2n}\, d\mathcal{L}^{2n-1}(w),$$

because the mapping ψ is smooth and therefore $\mathcal{H}^{2n-1}(\psi(S)) \le C_3 \mathcal{L}^{2n-1}(S)$, while the last integral obviously tends to zero as $r \to +0$. To the second integral in (2) we apply equality (1). Then

$$\lim_{r\to+0} \int\limits_{S_\psi(a,r)\cap D} U(\psi(\zeta)) = \lim_{r\to+0} \mu_a \int\limits_{S(0,r)\cap\psi(D)} U(w) = \mu_a \tau_\psi(a),$$

because

$$\int\limits_{S(0,r)\cap\psi(D)} U(w) = \frac{\mathcal{L}^{2n-1}[S(0,r)\cap\psi(D)]}{\mathcal{L}^{2n-1}[S(0,r)]}$$

by lemma 2.1 in [4]. □

Now assume that $\psi = (\psi_1, \dots, \psi_n)$ is a holomorphic mapping from $\mathbb{C}^n$ to $\mathbb{C}^n$, consisting of entire functions and possessing the unique zero which is the origin. The multiplicity of the zero of the mapping ψ is denoted by μ.

Let us define the integrals

$$\int_{\partial D_\zeta} f(\zeta)\, U(\psi(\zeta - z)) = \begin{cases} F^+(z), & z \in D; \\ F^-(z), & z \notin \overline{D}. \end{cases}$$

Lemma 2. *If the function f belongs to $C^\gamma(\partial D)$, $\gamma > 0$, then the integrals $F^\pm$ can be continuously extended to ∂D and $F^+(z) - F^-(z) = \mu f(z)$ on ∂D.*

Proof. Let us extend f into the neighborhood V of the boundary of the domain D up to a function satisfying the Hölder condition with exponent γ in this neighborhood. Let us prove that the functions of the form

$$\int_{\partial D_\zeta} (f(\zeta) - f(z)) U(\psi(\zeta - z))$$

are continuous in V. To this end we have to show that the integrals

$$\int_{S_\zeta} (f(\zeta) - f(z)) \frac{\overline{\psi_k(\zeta - z)}}{|\psi(\zeta - z)|^{2n}} d\overline{\psi}[k] \wedge d\psi$$

converge absolutely (here S is a certain neighborhood of the point z on the surface ∂D). Inequality (3) applied to $\psi(\zeta - z)$ and Hölder property of the function f give us the inequality

$$|f(\zeta) - f(z)| \le c|\zeta - z|^\gamma \le c_1 |\psi(\zeta - z)|^{\gamma\alpha}$$

for the points ζ from a sufficiently small neighborhood of z. Using (1) as in the proof of Lemma 1, we obtain

$$\int_{S_\zeta} |f(\zeta) - f(z)| \frac{|\psi_k(\zeta - z)|}{|\psi(\zeta - z)|^{2n}} |d\overline{\psi}[k] \wedge d\psi| \le c_1 \int_{S_\zeta} |\psi(\zeta - z)|^{\gamma\alpha+1-2n} |d\overline{\psi}[k] \wedge d\psi| \le$$

$$\le c_1 \mu \int_{\psi(S)} |w|^{\gamma\alpha+1-2n} d\mathcal{H}^{2n-1}(w) \le c_2 \int_S |w|^{\gamma\alpha+1-2n}\, d\mathcal{L}^{2n-1}(w).$$

Obviously, the last integral converges. The formula

$$\int_{\partial D} U(\psi(\zeta - z)) = \begin{cases} \mu, & z \in D; \\ 0, & z \notin \overline{D}, \end{cases}$$

concludes the proof of Lemma 2. □

Let us turn back to the initial mapping ψ.

Lemma 3. *For functions $f \in \mathcal{C}^\gamma(\partial D)$, $\gamma > 0$, the equality is valid*

$$\text{v. p.}^\psi \int_S f(\zeta)U(\psi(\zeta)) = \text{v. p.} \int_S f(\zeta)U(\psi(\zeta)).$$

This Lemma generalizes proposition in [3] about the equality of principal values for the case of simple zeros of the mapping ψ.

Proof. As has been shown in Lemma 2, the integral

$$\int_S (f(\zeta) - f(z))U(\psi(\zeta))$$

converges absolutely, therefore the principal values are equal to this integral. So, now we should prove only that

$$\text{v. p.}^\psi \int_S U(\psi(\zeta)) = \text{v. p.} \int_S U(\psi(\zeta)).$$

Let us transform the following integral (r is sufficiently small), taking $S = \partial D \cap B(z, R)$,

$$\int\limits_{S \setminus B_\psi(z,r)} U(\psi(\zeta)) = \int\limits_{\partial(D \cap B(z,R) \setminus B_\psi(z,r))} U(\psi(\zeta)) - \int\limits_{D \cap S(z,R)} U(\psi(\zeta)) + \int\limits_{D \cap S_\psi(z,r)} U(\psi(\zeta)) =$$
$$= - \int\limits_{D \cap S(z,R)} U(\psi(\zeta)) + \int\limits_{D \cap S_\psi(z,r)} U(\psi(\zeta))$$

by the formula for the multidimensional logarithmic residue. Then it remains to prove that

$$\lim_{r \to +0} \int\limits_{D \cap S_\psi(z,r)} U(\psi(\zeta)) = \lim_{r \to +0} \int\limits_{D \cap S(z,r)} U(\psi(\zeta)).$$

By virtue of theorem 3.2.5. in [2] (equality (1)), we have

$$\int\limits_{D \cap S_\psi(z,r)} U(\psi(\zeta)) = \mu_z \int\limits_{\psi(D) \cap S(0,r)} U(w), \quad \int\limits_{D \cap S(z,r)} U(\psi(\zeta)) = \mu_z \int\limits_{\psi(D \cap S(z,r))} U(w).$$

Consequently, we need to prove that

$$\lim_{r \to +0} \int\limits_{\psi(D) \cap S(0,r)} U(w) = \lim_{r \to +0} \int\limits_{\psi(D \cap S(z,r))} U(w).$$

In this equality we can take instead of $\psi(D)$ the tangent cone Π to $\psi(D)$ at the point 0. Let us show that

$$\int\limits_{\Pi \cap S(0,r_1)} U(w) = \int\limits_{\Pi \cap \psi(S(z,r_2))} U(w).$$

Let us consider a domain G bounded by the surfaces $\Pi \cap S(0, r_1)$, $\Pi \cap \psi(S(z, r_2))$ and by a part of the conic surface $M \cap \partial\Pi$ (r_1 and r_2 are chosen so that the ball $B(0, r_1)$ contains the surface $\psi(S(z, r_2))$). By the Bochner–Martinelli formula, we have

$$\int_{\partial G} U(w) = 0,$$

therefore

$$\int\limits_{\Pi\cap S(0,r_1)} U(w) - \int\limits_{\Pi\cap\psi(S(z,r_2))} U(w) = \int\limits_M U(w).$$

Let us show that

$$\int_M U(w) = 0.$$

We pass from the complex coordinates w to the real coordinates $w_j = \xi_j + i\xi_{n+j}$, $j = 1,\dots,n$. Then (see [3] or [4], § 20) we have

$$\operatorname{Re} U(w) = \frac{(n-1)!}{2\pi^n}\sum_{k=1}^{2n}(-1)^{k-1}\frac{\xi_k}{|\xi|^{2n}}d\xi[k],$$

$$\operatorname{Im} U(w) = -\frac{(n-2)!}{4\pi^n}d\Big(\sum_{k=1}^{n}\frac{1}{|\xi|^{2n-2}}d\xi[k,n+k]\Big), \quad n>1,$$

and with $n=1$ we have

$$\operatorname{Im} U(w) = -\frac{d\ln|\xi|^2}{4\pi}.$$

The restriction of the differential form $\operatorname{Re} U(w)$ to the conic surface M (at the points of the smoothness of M) is equal to zero. Indeed, let M be given by zeros of a homogeneous real-valued function φ: $M = \{\xi : \varphi(\xi) = 0\}$. Then at the points of the smoothness of M the restriction of the form $d\xi[k]$ to M is equal to $(-1)^{k-1}\gamma_k d\sigma$, where $\gamma_k = \frac{\partial\varphi}{\partial\xi_k}\frac{1}{|\operatorname{grad}\varphi|}$ are the direction cosines of normal, while $d\sigma$ is element of the surface M. Then

$$\sum_{k=1}^{2n}(-1)^{k-1}\frac{\xi_k}{|\xi|^{2n}}d\xi[k]\Big|_M = \sum_{k=1}^{2n}\xi_k\frac{\partial\varphi}{\partial\xi_k}\frac{1}{|\operatorname{grad}\varphi||\xi|^{2n}}d\sigma = l\varphi\frac{1}{|\operatorname{grad}\varphi|\,|\xi|^{2n}}d\sigma = 0$$

by the Euler formula on homogeneous functions (l is the degree of homogeneity of the function φ); the $(2n-1)$-dimensional measure of the set of non-smoothness points is equal to zero.

Integration by M will be carried out with the help of the real straight lines on M of the form

$$l_b = \{\xi : \xi_j = b_j t,\ j = 1,\dots,2n,\ t\in\mathbb{R}\},$$

where $|b| = 1$. For a fixed $b\in S(0,1)$, the variable t changes from a certain number $r_2(b)$ to r_1. The function $r_2(b)$ is measurable. Thus, M is a fiber bundle over the cycle $\partial\Pi\cap S(0,1)$. In these variables

$$\operatorname{Im} U(w) = c_n d\Big(\frac{dt}{t}\wedge\sum_{k,j}\pm b_k db[j,k,n+k]\Big) = c_n\frac{dt}{t}\wedge\sum_{k=1}^{n}db[k,n+k],$$

because the form containing the product of more than $2n-2$ differentials db_j vanishes on $S\cap\partial\Pi$. Then

$$\int_M \operatorname{Im} U(w) = c_n\int\limits_{S(0,1)\cap\partial\Pi}\ln\frac{r_1}{r_2(b)}\sum_{k=1}^{n}db[k,n+k].$$

In almost all points of $S\cap\partial\Pi$ the variables b_k, b_{n+k} are functions of other variables b_j, $j\neq k, n+k$. Therefore the last integral takes the form

$$\int\limits_{S(0,1)\cap\partial\Pi}\sum_{k=1}^{n}\ln\Phi_k(b_1,\dots,[k],\dots,[n+k],\dots,b_{2n})db[k,n+k] =$$

$$= \int\limits_{S(0,1)\cap\Pi} d\Big(\sum_{k=1}^{n}\ln\Phi_k(b_1,\dots,[k],\dots,[n+k],\dots,b_{2n})db[k,n+k]\Big) = 0$$

by the Stokes formula. □

The proof of Theorem follows from Lemmas 1 and 3.

Lemma 2 makes it possible to strengthen theorem 1 in [6], which was proved for smooth functions.

Corollary. Let D be a bounded domain in $\mathbb{C}^n$ with a connected smooth boundary. If for the function $f \in \mathcal{C}^\gamma(\partial D)$ the integral $F^-(z)$ is zero outside $\overline{D}$, then the function f can be holomorphically extended to the domain D.

The proof repeats that of theorem 1 in [6] with the use of Lemma 2 of this article instead of corollary 1 in [6].

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20 September 1999
Revised 21 December 2000

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