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The Description of Varieties of Rings whose Finite Rings are Uniquely Determined by their Zero-Divisor Graphs

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Abstract—The zero-divisor graph of an associative ring R is defined as follows. The vertices of the graph are all the nonzero elements of the ring. Two different vertices x and y of the graph are connected by an edge if and only if $xy = 0$ or $yx = 0$.
In this paper, we give the complete description of varieties of associative rings all of whose finite rings are uniquely determined by their zero-divisor graphs.

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1. In this paper, we consider associative rings (not necessarily commutative and not necessarily with identity).

Definition. The *zero-divisor graph of a ring R* is the graph $\Gamma(R)$ satisfying the conditions: the vertices of $\Gamma(R)$ are all the nonzero zero divisors of the ring (one-sided and two-sided), two different vertices x, y are connected by an edge if and only if $xy = 0$ or $yx = 0$.

The notion of zero-divisor graph was introduced in [1]. I. Beck introduced this notion for a *commutative* ring, and all the elements of the ring were taken as vertices of the graph. In [2], the definition was changed: as vertices of the zero-divisor graph of a commutative ring, the authors considered only nonzero zero divisors. Further the notion of zero-divisor graph has been extended to the case of *noncommutative rings* (see, e.g., [3]).

One can easily give examples of nonisomorphic rings with equal zero-divisor graphs. For example, if A is a countable-dimensional algebra with zero multiplication over a field $\mathbb{Z}_p$ and B is a countable-dimensional algebra with zero multiplication over a field $\mathbb{Z}_q$, where p and q are different prime numbers, then $\Gamma(A) \cong \Gamma(B)$, but $A \ncong B$. In other words, even in the variety var $\langle xy = 0\rangle$, there are examples of infinite nonisomorphic rings whose zero-divisor graphs have the same structure. In this connection, the following question is of interest: under which conditions the equality of the zero-divisor graphs implies the isomorphism of the rings. Certain results giving answer to this question for commutative rings were obtained in [4]. In the present paper, this problem is studied in terms of varieties, namely, we study varieties of associative rings in which each finite ring is uniquely determined by its zero-divisor graph. In other words, we study the properties of a variety of rings $\mathfrak{M}$ for which the equality $\Gamma(R) = \Gamma(S)$ for finite rings $R, S \in \mathfrak{M}$ implies $R \cong S$. Earlier, such varieties were studied in [5, 6], but the complete description has not been obtained. In the present paper, the varieties in which all finite rings are uniquely determined by their zero-divisor graphs are completely described.

2. Below we introduce the notation and the notions used in the paper.

A *complete n-vertex graph K_n* is a graph (without loops and multiple edges) with n vertices each two of which are adjacent.

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