Inverse Boundary-Value Problems of Cauchy Type for Harmonic Functions

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Abstract—We apply two methods for solving the inverse boundary-value problem (the so-called problem (A)) in the Cauchy statement for an analytic function and an unknown curve \( \Gamma \). We obtain criteria for \( \Gamma \) to be the unit circle. We apply the proposed methods for solving a modified Hadamard example and generalize the obtained results for the case of doubly connected domains.

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A. S. Demidov and D. A. Platuschchikhin [1] studied the Cauchy problem for the two-dimensional Laplace equation and obtained the function

\[
z(\zeta) = P_0 + \int_1^\zeta \exp(A + iB) \, d\zeta,
\]

which is analytic in the annular neighborhood \( E(r_1, r_2) = \{ \zeta : r_1 < |\zeta| < r_2 \}, \) \( r_1 < 1 < r_2 \), of the unit circle \( |\zeta| = 1 \) and maps this circle on a curve \( \Gamma \). The conditions of the Cauchy problem are presented in the form

\[
u|_\Gamma = f(s) = \Re \sum_{k \geq 0} f_k e^{iks}, \quad \frac{\partial u}{\partial \nu}|_\Gamma = g(s) = \Re \sum_{k \geq 1} g_k e^{iks},
\]

where \( s \) is the length of the arc (the arc coordinate) of the curve \( \Gamma \), \( 0 \leq s \leq 2\pi \), \( \nu \) is the outer normal at points of the curve \( \Gamma \) bounding a finite simply connected domain \( \Omega \). The curve \( \Gamma \) is determined in terms of the function \( Q(s) = N(s) - s \), where \( N(s) \) is the angle between the abscissa axis and the normal \( \nu \) at the point \( P_s \in \Gamma \); one can also use a natural equation of the curve \( \Gamma \) with \( |dz/d\zeta| = 1 \).

Problem (A) implies the determination of the curve \( \Gamma \) and the analytical function \( z(\zeta) \) in the form (1) from boundary data (2). It has a unique solution. We establish properties of functions \( f(s) \) and \( g(s) \) that turn \( \Gamma \) into the unit circle (Theorem 1), describe the process of solving problem (A) by the Demchenko scheme, propose a new approach to its solution (Theorem 2), and reduce problem (A) to the basic inverse boundary-value problem by means of the Tumashev–Nuzhin method (Theorem 3). We also solve a modified Hadamard example in various ways and present preliminary results on the generalization of problem (A) for the case of a doubly connected domain.

Note that the solution of the Cauchy problem after the construction of the curve \( \Gamma \) is \( u = \Re \tilde{f}[\zeta(z)] \); see below its special form (10) for the Hadamard example.

1. The uniqueness of a solution to problem (A). We say that two curves \( \Gamma \) and \( \Gamma_1 \) give one and the same solution to problem (A), if \( \Gamma_1 \) can be obtained from \( \Gamma \) by the rotation and translation. One can easily prove the following lemma.

**Lemma.** The solution to problem (A) is unique.

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