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CONDITIONAL EXPECTATIONS ON REAL W^*-ALGEBRAS AND JW-ALGEBRAS

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A conditional expectation E on a W^*-algebra is a linear positive normal map such that E is a projection (i.e., an idempotent map) with norm 1. In [1] Takesaki found the necessary and sufficient conditions for the existence of a conditional expectation on W^*-algebra which preserves normal semifinite weight. In [2]–[5] the conditional expectations on the Jordan operator algebras (JW-algebras) were defined, and relationship between them and conditional expectations on the enveloping W^*-algebras of these JW-algebras was studied. In [6] for JW-algebras a version of Takesaki's theorem on the existence of conditional expectations preserving faithful normal semifinite weight was proved, and then in [7] for JBW-algebras an analog of Takesaki's theorem on the existence of conditional expectations preserving a faithful normal linear functional of norm 1 (a state) was obtained.

The aim of present article is to obtain a necessary and sufficient condition for the existence of conditional expectation on a reversible JW-algebra which preserves a faithful normal semifinite weight.

1. *Preliminaries.* A JW-algebra is a real linear space A of selfadjoint operators in $B(H)$, which is complete with respect to the Jordan (symmetrized) multiplication $x \circ y = \frac{1}{2}(xy + yx)$, $x, y \in A$, and is closed with respect to the weak operator topology. A JW-algebra A is said to be reversible if $x_1 x_2 \dots x_n + x_n x_{n-1} \dots x_1 \in A$ for all $x_1, x_2, \dots, x_n \in A$. Any reversible JW-algebra A can be characterized by the property $A = R(A)_{sa} = \{x \in R(A) : x = x^*\}$, where $R(A)$ is the minimal weakly closed real subalgebra of $B(H)$ which contains A. A reversible JW-algebra A can be decomposed with respect to the center into the direct sum of JW-algebras A_r and A_c, $A = A_r \oplus A_c$, where A_c is the set of self-adjoint operators of W^*-algebra $R(A_c) = \mathcal{U}(A_c)$, and A_r is the set of self-adjoint operators of real W^*-algebra $R(A_r)$.

Recall that a real W^*-algebra is a real $*$-algebra R of bounded linear operators in a complex Hilbert space H, such that R has unit, R is closed with respect to the weak operator topology, and $R \cap iR = \{0\}$. The enveloping W^*-algebra $\mathcal{U}(R)$ of R (i.e., the minimal W^*-algebra containing R) has the form $\mathcal{U}(R) = R + iR$, and R defines an involutive $*$-anti-automorphism α_R on $\mathcal{U}(R)$, $\alpha_R(x + iy) = x^* + iy^*$, where $x, y \in R$. Respectively, R is determined by α_R, $R = \{x \in \mathcal{U}(R) : \alpha_R(x) = x^*\}$.

For the theory of JW-algebras we refer the reader to [8]–[10], and for the theory of real W^*-algebras to [11].

2. *Conditional expectations on real W^*-algebras.* Let us define a conditional expectation on a real W^*-algebra with respect to its real W^*-subalgebra.

Definition. Let R be a real W^*-algebra, and Q a real W^*-subalgebra in R. An injective linear map $E : R \to Q$ is called a conditional expectation if E is a positive projection with norm 1. E is