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INVERTIBILITY OF LINEAR DIFFERENCE OPERATORS WITH CONSTANT COEFFICIENTS

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Let X and Y be complex Banach spaces, $\mathrm{Hom}(X,Y)$ a Banach space of linear bounded operators (homomorphisms) acting from X with values in Y, $\mathrm{End}\, X = \mathrm{Hom}(X,X)$ a Banach endomorphism algebra of the space X. We denote by I the identity operator in any of the spaces under consideration. We denote by $\sigma(A)$, $\rho(A)$, and $r(A)$ the spectrum, resolvent set, and spectral radius, respectively, of the linear operator A.

Let G be a locally compact Abelian group, $\widehat{G}$ the dual group of continuous unitary characters of the group G (to write the algebraic operation we use the additive form in the first group and multiplicative in the second group). We denote by $\mathcal{F}(G,X)$ (in what follows, $\mathcal{F}$) one of the following Banach spaces of functions (Bochner-)measurable and taking values in X.

$L_p = L_p(G,X)$ is the space of functions integrable on G (with respect to the Haar measure on G) in power $p \in [1,\infty)$, $L_\infty = L_\infty(G,X)$ the space of essentially bounded on G functions, $C = C(G,X)$ the subspace of continuous functions from $L_\infty(G,X)$, $C_0 = C_0(G,X)$ the subspace of functions from $C(G,X)$, which converge to zero at infinity.

If $G = \mathbb{Z}$ is the group of integer numbers, then the dual group $\widehat{G}$ is identified with (is isomorphic to) the circle $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ from the field of complex numbers $\mathbb{C}$. In this case instead of the symbol $L_p(\mathbb{Z},X)$ the symbol $l_p(\mathbb{Z},X)$ is often used. For the corresponding Banach spaces of one-sided sequences we use the notation $\mathcal{F}(\mathbb{Z}_+,X)$, where $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, and, in particular, $l_p(\mathbb{Z}_+,X)$. In addition, we assume $c_0(\mathbb{Z}_+,X) = C_0(\mathbb{Z}_+,X)$, $c_0(\mathbb{Z},X) = C_0(\mathbb{Z},X)$.

Let A, B be two operators from $\mathrm{Hom}(X,Y)$. By these operators we define the difference operators $\mathcal{D}$, $\mathcal{D}_+$, $\mathcal{D}_+^0$ by the following formulas:

$$(\mathcal{D}x)(g) = Ax(g) - Bx(g-g_0), \quad g \in G, \quad g_0 \in G, \quad x \in \mathcal{F}(G,X),$$
$$(\mathcal{D}_+x)(k) = Ax(k) - Bx(k+1), \quad k \in \mathbb{Z}_+, \quad x \in \mathcal{F}(\mathbb{Z}_+,X),$$
$$(\mathcal{D}_+^0x)(k) = \begin{cases} Ax(0), & k = 0; \\ Ax(k) - Bx(k-1), & k \geq 1, \quad x \in \mathcal{F}(\mathbb{Z}_+,X). \end{cases}$$

The operator $\mathcal{D}$ belongs to the space $\mathrm{Hom}(\mathcal{F}(G,X),\mathcal{F}(G,Y))$, while the operators $\mathcal{D}_+$ and $\mathcal{D}_+^0$ to the space $\mathrm{Hom}(\mathcal{F}(\mathbb{Z}_+,X),\mathcal{F}(\mathbb{Z}_+,Y))$.

In this article we obtain the necessary and sufficient conditions for invertibility of such difference operators and also formulas for the inverse operators. We consider the conditions for existence and construction of the left and right inverse operators to the difference operators; we also describe both the kernels and images.

Investigation of the conditions for invertibility of the difference operators under consideration is of interest in view of its close connection with invertibility of abstract parabolic operators of the

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