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Matrix Bernoulli Equations. I

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We establish sufficient conditions for solvability by quadratures of J. Bernoulli's equation given over the set of square matrices.

1. Statement of the problem. Let M_n be the set of $n \times n$-matrices with unity E over the field of real numbers R, $L_{N(n)}$ a representation in M_n of an N-dimensional ($N \leq n^2$) Lie algebra with basis elements E_α and structure constants $C^\gamma_{\alpha\beta}$ ($\alpha, \beta, \gamma = \overline{1, N}$), $M^k_n(t)$ the set of matrices whose components are functions of the kth continuity class with respect to a real variable t, $L^k_{N(n)}(t) \equiv L_{N(n)} \cap M^k_n(t)$, and E_{ij} the basis of the full matrix algebra M_n.

As is known, many real phenomena are described and studied with the use of ordinary differential equations (ODE) over varieties with various algebraic structures. Linearization of dynamical equations describing such processes often leads to significant errors. These errors can be avoided by means of complication of the ODEs, in particular by the use of nonlinear equations.

However, by now there is a very narrow class of nonlinear differential equations over noncommutative manifolds that can be used for modeling of real processes and have sufficiently wide conditions of exact solvability (by quadratures). A generalization to the matrix set of J. Bernoulli's equation

$$x' \equiv \frac{d}{dt}x = a(t)x + b(t)x^m, \quad (a(t), b(t)) \subset C^0(t), \;\; b(t) \not\equiv 0, \;\; m \neq 0, 1, \tag{1}$$

known in the theory of scalar ODEs allows ones to enlarge this class.

Since, by I. D. Ado's theorem [1], any Lie algebra has a matrix representation, for modeling of processes given over complicated algebraic varieties, one can use matrix ODEs (MODEs). Among these are matrix Bernoulli equations (MBE), i.e., equations of the form

$$X' = A_1 X + X A_2 + X B_1 X \cdots X B_{m-1} X, \quad (A_k, B_l) \subset M^0_n(t), \;\; 0 \neq X \in M^1_n(t), \tag{2}$$

where the natural number m does not equal to 0 and 1. Since the above equation is equivalent to a normal system of scalar equations whose right-hand side satisfies the conditions of G. Peano's theorem, an MBE always has a solution.

As is known, the solution of Eq. (1) can be expressed in terms of radicals. Naturally, radicals are used in the solution of Eq. (2) generalizing Eq. (1). Note that, over an associative algebra, $X = \sqrt[m]{A}$ is a solution of the equation $X^m = A$. In particular, for the values t preserving the Jordan form of the matrix $A(t)$, $\sqrt[m]{A(t)}$ is defined in [2] (P. 212–219).

In contrast to the classical case, an MBE cannot be integrated in the general form. The reason is that the algebra M_n is noncommutative, and, over M_n, even first-order linear ODEs (LODE.1s), which are a special case of Eq. (2), cannot be solvable by quadratures. For such equations, there are only sufficient integrability conditions depending on the algebraic structure of the subalgebra of M_n containing parameters of the LODE [3].

In this paper, we pose the problem of finding sufficient conditions for solvability by quadratures of a matrix J. Bernoulli's equation in the form (2) with parameters A_k and B_l restricted in the Euclidean norm.