Graphs and algorithms in direct decomposition theory of torsion-free abelian groups

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$$CD_n \subset ACD_n \subset TFAG_n$$

$X \in TFAG_n$ if $\mathbb{Z}^n \subset X \subset \mathbb{Q}^n = \mathbb{Q} \oplus \mathbb{Q} \ldots \oplus \mathbb{Q}$
(X is a torsion-free abelian group of rank n)

$A \in CD_n$ if $A = A_1 \oplus A_2 \ldots \oplus A_n$ with $A_i \in \mathbb{Q}$
(A is a completely decomposable abelian group of rank n)

$X \in ACD_n$ if there exists $A \in CD_n$ such that $A \subset X$ and $|X/A| < \infty$
(X is an almost completely decomposable abelian group of rank n)

Main Definitions

A torsion-free abelian group is completely decomposable, a CD-group, if it is a direct sum of subgroups of the rationals $\mathbb{Q}$.

An ACD-group X (almost completely decomposable group) is a torsion-free abelian group of finite rank, that contains a fully invariant completely decomposable group $A = R(X)$, its regulator, for which X/A is a finite group.

In case of cyclic X/A we say that ACD-group X is a CRQ-group (with cyclic regulator quotient).

$$CD_n \subset CRQ_n \subset ACD_n \subset TFAG_n$$

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E. Blagoveshchenskaya. Almost completely decomposable abelian groups and their endomorphism rings, Mathematics in Polytechnical University, St. Petersburg, 2009.

Example. Let $\tau_p = \langle \frac{1}{p}, \frac{1}{p^2}, \ldots, \frac{1}{p^k}, \ldots \rangle \subset \mathbb{Q}$ for any prime p and $\tau_{p_1} \oplus \tau_{p_2}$ be a completely decomposable group, $p_1 \neq p_2$.

Let $X \cong \langle \tau_{p_1} \oplus \tau_{p_2}, b \rangle \subset \mathbb{Q}^2$ with $b = \frac{(1,1)}{q}$ and q, p_1, p_2 be different primes. Almost completely decomposable group X is indecomposable.

Let $Y \cong \langle \tau_{p_1} \oplus \tau_{p_2}, c \rangle \subset \mathbb{Q}^2$, with $c = \frac{(1,1)}{h}$ and h, q, p_1 and p_2 be different primes. Let $G = X \oplus Y \subset \mathbb{Q}^4$.

Regulator: $R(G) \cong \tau_{p_1} \oplus \tau_{p_1} \oplus \tau_{p_2} \oplus \tau_{p_2}$, $G/R(G) \cong \mathbb{Z}/q\mathbb{Z} \oplus \mathbb{Z}/h\mathbb{Z}$.

$G \cong \langle \tau_{p_1} \oplus \tau_{p_2}, b \rangle \oplus \langle \tau_{p_1} \oplus \tau_{p_2}, c \rangle \cong \langle \tau_{p_1} \oplus \tau_{p_2}, f \rangle \oplus \tau_{p_1} \oplus \tau_{p_2}$ with $f = \frac{(1,1)}{qh}$ as $qh | f = h(qb) + q(hc)$ and $f \in R(G)$:

$$4 = 2 + 2 = 2 + 1 + 1$$

B. Jonsson (1957,1959), A.L.S. Corner (1961, 1969)

$X = X_1 \oplus \ldots \oplus X_s = Y_1 \oplus \ldots \oplus Y_t, \;\; s \neq t.$

$\mathrm{rk}\,(X) = \mathrm{rk}\,(X_1) + \ldots + \mathrm{rk}\,(X_s) = \mathrm{rk}\,(Y_1) + \ldots + \mathrm{rk}\,(Y_t)$

L. Fuchs, Problem 68: What partitions of a natural number $n = m_1 + m_2 + \ldots + m_s = l_1 + l_2 + \ldots + l_t$ can be associated with two different direct decompositions of a torsion-free abelian group X of rank n into indecomposable summands of ranks $m_1, m_2, \ldots, m_s$ and $l_1, l_2, \ldots, l_t$?

E. Blagoveshchenskaya (1983):
$X \in CRQ$, $m_i \leq n - t + 1$ and $l_j \leq n - s + 1$

Theorem 0.1 Let $n > r \geq 2$ be natural numbers. Then there exists a block-rigid crq-group X of rank n decomposing into direct sums of indecomposable summands of ranks $r_1, r_2, \ldots, r_s$ for any natural $s:\ 1 < s < n$ and all possible partitions $n = r_1 + r_2 + \ldots + r_s$ with $r = r_1 \geq r_2 \geq \ldots \geq r_s \geq 1$.

Lemma 0.2 Let $n > r \geq 2$ be natural numbers. Then there exists an r-colorable graph Γ with n vertices such that for any natural $s:\ 1 < s < n$ and any possible partition of n into s natural summands $n = r_1 + r_2 + \ldots + r_s$ with $r = r_1 \geq r_2 \geq \ldots \geq r_s \geq 1$ there exists an admissible transformation of Γ leading to the set of connection components $L_1, L_2, \ldots, L_s$ with $r_1, r_2, \ldots, r_s$ vertices accordingly.

Fix an arbitrary partition $n = r_1 + r_2 + \ldots + r_s$ with $r = r_1 \geq r_2 \geq \ldots \geq r_s \geq 1$. Let us take a "preparatory"connected graph $\mathcal{G}$ with n vertices marked by natural numbers $1, 2, \ldots, n$ factorized modulo r. Assume that the vertices, which are congruent modulo r, have the same color. We also assume that graph $\mathcal{G}$ is connected by the edges $p_1, p_2, \ldots, p_{n-1}$ so that each p_i join the vertices i and $i+1$ ($\mathcal{G}$ is the simplest case of trees).

Definition   Let G and H be torsion-free abelian groups of finite rank. Then G and H are called nearly isomorphic (in symbols $G \cong_{nr} H$) if and only if for any prime q there are monomorphisms $\eta_q : G \longrightarrow H$ and $\xi_q : H \longrightarrow G$ such that $H/\eta_q(G)$ and $G/\xi_q(H)$ are finite groups and $|H/\eta_q(G)|$ and q as well as $|G/\xi_q(H)|$ and q are relatively prime.

Arnold (1982):   If $G \cong_{nr} H$ and $G = G_1 \oplus \ldots \oplus G_k$ then there exists a decomposition $H = H_1 \oplus \ldots \oplus H_k$ such that
$G_1 \cong_{nr} H_1$, ... , $G_k \cong_{nr} H_k$.

E. Blagoveshchenskaya, A. Mader (1994):

Classsification of CRQ-groups up to near-isomorphism

E. Blagoveshchenskaya (2004): Classification of CRQ-groups up to isomorphism.

Primary factor representations of X and $\mathrm{End}\,(X)$ accordingly: $e = \prod_{p \in P} p^{\gamma_p}$

$$X = \sum_{p \in P} X_p \quad \text{with} \quad A \subset X_p \subset \frac{A}{p^{\gamma_p}}$$

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$$\mathrm{End}\,(X) = \bigcap_{p \in P} \mathrm{End}\,(X_p) \quad \text{with} \quad p^{\gamma_p}\mathrm{End}\,(A) \subset \mathrm{End}\,(X_p) \subset \mathrm{End}\,(A)$$

Let B(P) be Boolean algebra of all subsets of P. The set $\{X_p :\ p \in P\}$ is given, $A = R(X_p)$.

Let $X(P') = \sum_{p \in P'} X_p$ and $E(P') = \mathrm{End}\,(X(P'))$ for any $P' \in B(P)$.

B(P)	B	B^*
P'	$X(P')$	$E(P')$
$P' \cup P''$	$X(P') + X(P'')$	$E(P') \cap E(P'')$
$P' \cap P''$	$X(P') \cap X(P'')$	$E(P') + E(P'')$
P	1	O
$\emptyset$	O	1

B and B^* are anti-isomorphic Boolean algebras.

E. Blagoveshchenskaya (2006)

Dualities of ACD additive structures (groups and rings)

$$A \subset \qquad X = \sum_{p\in P} X_p = \bigcap_{p\in P} \frac{X_p}{e'_p} \qquad \subset \frac{A}{e}$$

$$e\mathrm{End}\,(A) \subset \qquad \mathrm{End}\,(X) = \bigcap_{p\in P} \mathrm{End}\,(X_p) = \sum_{p\in P} e'_p \mathrm{End}\,(X_p) \qquad \subset \mathrm{End}\,(A)$$

with $e'_p = \frac{e}{p^{\gamma_p}}$, $\quad e = \prod_{p\in P} p^{\gamma_p}$.

Recall $p^{\gamma_p}\mathrm{End}\,(A) \subset \mathrm{End}\,(X_p) \subset \mathrm{End}\,(A)$.

Blagoveshchenskaya (2004-2006)

Theorem 1 Any automorphism $\phi \in \mathrm{Aut}\,(\mathrm{End}\,X_p)$ can be uniquely extended to an automorphism $\Phi \in \mathrm{Aut}\,(\mathrm{End}\,A)$,
or in other words: $\mathrm{Aut}\,(\mathrm{End}\,X_p) \subset \mathrm{Aut}\,(\mathrm{End}\,A)$.

Theorem 2 $\mathrm{Aut}\,(\mathrm{End}\,(X)) = \bigcap_{p\in P} \mathrm{Aut}\,(\mathrm{End}\,(X_p))$

Baer (1943), Kaplansky (1952):

If X and Y are torsion groups and $\mathrm{End}\,X \cong \mathrm{End}\,Y$ then for any isomorphism $\Psi : \mathrm{End}\,(X) \longrightarrow \mathrm{End}\,(Y)$ there exists an isomorphism $\phi : X \longrightarrow Y$ such that $\eta\Psi = \phi^{-1}\eta\phi$ for each $\eta \in \mathrm{End}\,(X)$.

Sebeldin (1972), Blagoveshchenskaya, Ivanov, Schultz (2001):

Let X and Y be completely decomposable groups of ring type. If $\Theta : \mathrm{End}\,(X) \to \mathrm{End}\,Y$ is a ring isomorphism, then there exists a group isomorphism $\theta : X \to Y$ such that for all $f \in \mathrm{End}\,(X)$, $f\Theta = \theta^{-1} f \theta$.

E. Blagoveshchenskaya (2004):

Let $X, Y \in ACD$ and $X \cong_{nr} Y$. Then $\mathrm{End}\,(X)^+ \cong_{nr} \mathrm{End}\,(Y)^+$.

In other words, if $X, Y \in ACD$ then

$e\mathrm{End}\,(A) \subset \mathrm{End}\,(X) \subset \mathrm{End}\,(A)$ and $e\mathrm{End}\,(A) \subset \mathrm{End}\,(Y) \subset \mathrm{End}\,(A)$ with $eX \subset A$ and $eY \subset A$.

Let $\overline{}: \mathrm{End}\,(A) \longmapsto \mathrm{End}\,(A)/e\mathrm{End}\,(A) = \overline{\mathrm{End}\,(A)}$.

Clearly $\overline{\mathrm{End}\,(X)} \subset \overline{\mathrm{End}\,(A)}$ and $\overline{\mathrm{End}\,(Y)} \subset \overline{\mathrm{End}\,(A)}$.

$\mathrm{End}\,(X) \cong_{nr} \mathrm{End}\,(Y)$ if and only if there exists a ring automorphism $\phi: \overline{\mathrm{End}\,(A)} \longrightarrow \overline{\mathrm{End}\,(A)}$ such that $\phi: \overline{\mathrm{End}\,(X)} \longrightarrow \overline{\mathrm{End}\,(Y)}$ is a ring isomorphism.

E. Blagoveshchenskaya, G. Ivanov, P. Schultz (2001):

Theorem in the near Baer-Kaplansky form

Let $X, Y \in CRQ$. Then $\mathrm{End}\,(X) \cong \mathrm{End}\,(Y)$ if and only if $X \cong_{nr} Y$.