ISSN 1066-369X, Russian Mathematics (Iz. VUZ), 2007, Vol. 51, No. 12, pp. 52–70. © Allerton Press, Inc., 2007.

*

The Structure of the Euler–Lagrange Mapping

D. Krupka[1]

[1] *Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic;*
La Trobe University, Bundoora, Victoria 3086, Australia[1]

Received April 12, 2007

Abstract—The purpose of this paper is to review properties of the Euler–Lagrange mapping in the higher order variational theory on fibred manifolds. We present basic theorems on the kernel of the Euler–Lagrange mapping, describing variationally trivial Lagrangians, and its image, characterizing variational source forms. We discuss invariance properties of Lagrangians and Euler–Lagrange forms, and the Noether's theory.

DOI: 10.3103/S1066369X07120043

1. INTRODUCTION

In this paper we present a survey of general properties of the Euler–Lagrange mapping of the higher order integral variational theory on fibred manifolds. The subject has been developed for a long time with various ideas; we discuss the kernel of the Euler–Lagrange mapping (*variationally trivial*, or *null* Lagrangians), its *image* (the *inverse problem* of the calculus of variations), and *invariance properties* (the *Noether theory*).

Throughout this paper, Y is a fibred manifold with base X and projection π, J^rY is the r-jet prolongation of Y, and $\pi^{r,s} : J^rY \to J^sY$, $\pi^r : J^rY \to X$ are the canonical jet projections. We denote $n = \dim X$, and $m = \dim Y - n$. For any open set $W \subset Y$ we denote $W^r = (\pi^{r,0})^{-1}(W)$. We use standard notation conventions for charts: the chart on J^rY, associated with a fibred chart (V, ψ), $\psi = (x^i, y^\sigma)$, on Y, where $1 \le n \le m$ and $1 \le \sigma \le m$, is denoted by (V^r, ψ^r), $\psi^r = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \dots, y^\sigma_{j_1 j_2 \dots j_r})$. $\Omega^r W$ is the exterior algebra of forms on W^r; $\Omega^r_p W$ ($\Omega^r_{p,X} W$ and $\Omega^r_{p,Y} W$) denotes the module of p-forms (π^r-horizontal and $\pi^{r,0}$-horizontal forms).

Let W be an open subset of Y, λ be a Lagrangian of order r for Y, defined on W^r, and E_λ be the Euler–Lagrange form of λ; E_λ is defined on the set $W^{2r} \subset J^{2r}Y$. Recall that if in a fibred chart (V, ψ), $\psi = (x^i, y^\sigma)$, on Y, λ has an expression

$$\lambda = \mathcal{L}\omega_0, \tag{1.1}$$

where the *Lagrange function* $\mathcal{L}$ depends on $x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \dots, y^\sigma_{j_1 j_2 \dots j_r}$, then

$$E_\lambda = E_\sigma(\mathcal{L})\omega^\sigma \wedge \omega_0, \tag{1.2}$$

where $\omega^\sigma = dy^\sigma - y^\sigma_j dx^j$, and the components $E_\sigma(\mathcal{L})$ are the *Euler–Lagrange expressions*, defined by

$$E_\sigma(\mathcal{L}) = \sum_{l=0}^{r} (-1)^l d_{p_1} d_{p_2} \dots d_{p_l} \frac{\partial \mathcal{L}}{\partial y^\sigma_{p_1 p_2 \dots p_l}}. \tag{1.3}$$

*The text was submitted by the author in English.

[1] E-mail: e-mailkrupka@@inf.upol.cz.