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One Generalization of the Class of Helical Functions

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Abstract—This paper is devoted to the study of domains whose boundaries are attainable by one-parameter families of curves formed by the rotation of a curve specially chosen for every family. We establish characteristics of analytic functions that map the unit circle on these domains. In addition, we single out subclasses of domains with rectifiable quasiconformal boundaries. We establish certain sufficient conditions for the univalence of functions that are analytic in mentioned domains.

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It is well-known [1, 2] that a schlicht domain G is helical of order δ, $\delta \in (-\pi/2, \pi/2)$, with respect to the point $z_0 = 0 \in G$ if a part of the logarithmic spiral with the equation $\mathrm{Im}(e^{i\delta} \ln z) = \mathrm{const}$ that connects any point $z \in G$ with z_0 entirely belongs to G.

Let a function $z = z(\zeta)$, $z(0) = 0$, be regular in the circle $E = \{\zeta : |\zeta| < 1\}$ and map E onto some domain G. The domain G is spiral with respect to the origin of coordinates if and only if

$$\mathrm{Re}\left[e^{i\delta}\frac{\zeta z'(\zeta)}{z(\zeta)}\right] > 0, \quad \zeta \in E, \quad |\delta| < \pi/2. \tag{1}$$

Regular in E functions $z = z(\zeta)$, $z(0) = 0$, which satisfy condition (1) are called helical (with respect to 0), and the class of these functions is denoted by S^*_δ.

In this paper we generalize the class of helical functions as follows: Instead of logarithmic spirals used as a base for constructing the class S^*_δ, we use spirals in a more general form. We also establish some properties of the mentioned classes of functions and domains. Moreover, we obtain conditions for the univalence of functions which are regular in domains under consideration.

1. THE CLASS OF FUNCTIONS D_φ

We say that a regular in E function $z = z(\zeta)$, $z(0) = 0$, $z'(0) = 1$, belongs to the class D_φ (cf. [3]) if it satisfies the condition

$$\mathrm{Re}\left\{\left[1 - i|z(\zeta)|\varphi'(|z(\zeta)|)\right]\zeta\frac{z'(\zeta)}{z(\zeta)}\right\} > 0, \quad \zeta \in E, \tag{2}$$

where $\varphi(r)$ is a real function defined and differentiable on $(0, \infty)$. If a continuous function $\varphi(r)$ is not differentiable at a finite number of points $r = r_j$, $j = 1, 2, \ldots, n$, then we assume that condition (2) is fulfilled at points ζ, where $|z(\zeta)|$ differs from points of non-differentiability of the function $\varphi(r)$. At points $\zeta \in E$, where $|z(\zeta)| = r_j$, $1 \le j \le n$, we require that

$$\mathrm{Re}\left[\zeta\frac{z'(\zeta)}{z(\zeta)}\right] > 0. \tag{3}$$

Let us prove the univalence of functions in the class D_φ. The following theorem is valid.

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