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The Resolvent Structure of a Volterra Equation with Nonsummable Difference Kernel

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Abstract—In this paper we study the asymptotic behavior of the resolvent of a Volterra linear integral equation whose difference kernel is nonsummable. For a certain class of such kernels the equation is reducible to an equation whose difference kernel is summable. This enables one to use the well-known results on the structure of resolvents of summable kernels in the case of a nonsummable kernel. We apply the obtained results to homogeneous kernels of degree -1.

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I. Consider the system

$$x(t) = \int_0^t K(t-s)x(s)ds + f(t), \quad t \geq 0, \tag{1}$$

where $f \in \mathbf{C}\big([0,\infty) \to \mathbb{C}^n\big)$, K is a complex-valued $n \times n$-matrix, and $K \in L[0,a]$ with any $a > 0$.

We set

$$K * x(t) = \int_0^t K(t-s)x(s)ds, \quad \widehat{K}(z) = \int_0^\infty e^{-zt}K(t)dt,$$

$\widetilde{K}x(t) = K * x(t)$ is an operator acting in $\mathbf{C}\big([0,\infty) \to \mathbb{C}^n\big)$, I is the identity operator or matrix, and δ is the unit element of the convolution algebra. Hereinafter P_r (or just P) stands for a polynomial of degree $\leq r$ independently of its coefficients and variable.

For $K \in L_1[0,\infty)$ the structure of the resolvent R_K of the kernel K and the asymptotic behavior of a solution to Eq. (1) are thoroughly studied (e.g., [1–8]).

One can study kernels $K \notin L_1[0,\infty)$ using the idea described in papers [3, 9, 10]. Namely, one can change the variable $x = (I - \widetilde{Q})y$ so as to make the kernel G defined by the equality

$$I - \widetilde{G} = (I - \widetilde{K})(I - \widetilde{Q}), \tag{2}$$

i.e., the kernel $G(t) = K(t) + Q(t) - K * Q(t)$, summable on $[0,\infty)$. Then $\widetilde{R}_K = -\widetilde{Q} + (I - \widetilde{Q})\widetilde{R}_G$, and, since the structure of R_G is studied sufficiently well, one can also define the structure of R_K.

Generally speaking, a kernel Q such that $G \in L_1[0,\infty)$ always exists. If we put $Q = -R_K$, then we get $G \equiv 0 \in L_1[0,\infty)$. However, in order to determine the structure of R_K we need certain properties of Q, for example, $Q \in L_1[0,\infty)$. But then Q such that $G \in L_1[0,\infty)$ for an arbitrary kernel K does not necessarily exist. In order to state requirements to the kernel K, note that

$$K(t) = G(t) - R_Q(t) + G * R_Q(t).$$

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