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Continuity and Compactness of Singular Integral Operators

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Abstract—In the space of square integrable functions we establish effective sufficient continuity and compactness conditions for singular integral operators with Cauchy kernels on a segment of the real axis.

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In studying direct and projective methods for solving complete singular integral equations [1–3]

$$Ax \equiv a(t)x(t) + \frac{1}{\pi}\int_{-1}^{+1} \frac{h(t,\tau)x(\tau)\,d\tau}{\tau - t} = y(t), \quad -1 \le t \le 1, \tag{0.1}$$

the following problem arises: Justify the continuity and compactness of singular integral operators in functional spaces; here $a(t) \in C[-1,1]$, $h(t,\tau) \in C[-1,1]^2$, and $y(t) \in L_2(-1,1)$ are known functions, $x(t) \in L_2(-1,1)$ is the desired function, the singular integral

$$S_0\varphi = S_0(\varphi;t) = \frac{1}{\pi}\int_{-1}^{+1} \frac{\varphi(\tau)\,d\tau}{\tau - t}, \quad -1 \le t \le 1, \ \ \varphi \in L_2(-1,1), \tag{0.2}$$

is understood as the Cauchy–Lebesgue principal value

$$S_0(\varphi;t) = \frac{1}{\pi}\lim_{\varepsilon\to+0}\left\{\int_{-1}^{t-\varepsilon} + \int_{t+\varepsilon}^{+1}\right\}\frac{\varphi(\tau)\,d\tau}{\tau - t}, \tag{0.3}$$

with $\varphi \in L_2(-1,1)$ the partial integrals

$$\int_{-1}^{t-\varepsilon} \frac{\varphi(\tau)\,d\tau}{\tau - t}, \quad \int_{t+\varepsilon}^{+1} \frac{\varphi(\tau)\,d\tau}{\tau - t}$$

are understood in the Lebesgue sense, and with $\varphi \in H_\lambda[-1,1]$, $0 < \lambda \le 1$, they are understood in the Riemann sense.

This work is devoted to solving the above problem on the basis of results of the theory of functions and approximations [4–6], functional analysis [7, 8], and the theory of singular integral equations [1–3].

Let us introduce the following spaces and classes of functions:

$C = C[-1,1]$ is the space of continuous on $[-1,1]$ functions $f = f(t)$ with the norm

$$\|f\|_C = \max_{-1\le t\le 1} |f(t)|, \quad f \in C;$$

$C[-1,1]^2$ is the space of continuous on $[-1,1]^2 = [-1,1;-1,1]$ functions $f = f(t,\tau)$ with an analogous norm;

$L_2 = L_2(-1,1)$ is the space of real functions $f = f(t)$ square integrable on $[-1,1]$ in the Lebesgue sense, where the scalar product and the norm are, respectively,

$$(f,g) = \int_{-1}^{+1} f(t)g(t)\,dt, \quad \|f\|_{L_2} = \left\{\int_{-1}^{+1} |f(t)|^2 dt\right\}^{1/2}, \ \ f,g \in L_2;$$

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