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ON THE CLASSIFICATION OF ALMOST PRODUCT STRUCTURES ON A MANIFOLD WITH LINEAR CONNECTION

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In his listing the most important G-structures on an m-dimensional differentiable manifold M, S. Kobayashi (see [1], Chap. 1, § 2) pointed out those with the group $G = GL(n,\mathbf{R}) \times GL(m-n,\mathbf{R})$ consisting of block-diagonal matrices $A = \operatorname{diag}\{B, C\}$, where $B \in GL(n,\mathbf{R})$ and $C \in GL(m-n,\mathbf{R})$. A natural bijective correspondence exists between these structures and the pairs $(\mathcal{H},\mathcal{V})$ of complementary n- and $(m-n)$-dimensional differentiable distributions on M.

The structures indicated above were the subject of study in various investigations (see, e. g., the survey [2]); special chapters or sections of the monographs [3], [4] and others were devoted to the theory of these structures. A.P. Norden (see [5], pp. 128–129) gave a partial classification of $GL(n,\mathbf{R}) \times GL(m-n,\mathbf{R})$-structures. In the base of his classification one sees the concept of parallel displacement of the structure distributions $\mathcal{H}$ and $\mathcal{V}$. In the present article, which continues that line of investigations, we reduce the problem of classification to the classical problem in the elementary theory of representations of the general linear group concerning decomposition of the tensor product of representations into irreducible components. As a result, along with the well-known classes of $GL(n,\mathbf{R}) \times GL(m-n,\mathbf{R})$-structures (see [5], pp. 128–129) new classes are singled out and geometrically described.

1. $GL(n,\mathbf{R}) \times GL(m-n,\mathbf{R})$-structures on an m-dimensional manifold

Let M be an m-dimensional manifold, $L(M)$ be the linear frame bundle over M with the structure group $GL(m,\mathbf{R})$. Consider the subgroup $G = GL(n,\mathbf{R}) \times GL(m-n,\mathbf{R})$ in $GL(m,\mathbf{R})$. We shall regard it as a group of linear transformations of the vector space $E = T_xM$, which leave invariant two complementary subspaces H and V of dimensions n and $m-n$, respectively ($m < n$).

Let a principal G-subbundle of $L(M)$ with structure group $G = GL(n,\mathbf{R}) \times GL(m-n,\mathbf{R})$ be given. This subbundle determines (see [1], p. 22) the two complementary differentiable distributions $\mathcal{H}$ and $\mathcal{V}$ on M of the dimensions n and $m-n$, respectively. $\mathcal{H}$ is called *the horizontal* distribution, while $\mathcal{V}$ — *the vertical* one.

A G-structure under consideration is said to be *semi-integrable* if one of its structure distributions, say $\mathcal{V}$, is involutive. In this case, through any point $x \in M$ a unique maximal integral manifold M' of $\mathcal{V}$ passes, and, in a neighborhood U of x, a local coordinate system $x^1, \dots, x^m$ exists such that $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}\}$ is a local basis of the distribution $\mathcal{V}$.

In accordance with the general theory (see [1], Chap. 1, § 1, p. 8, and § 2, p. 22), if a $GL(n,\mathbf{R}) \times GL(m-n,\mathbf{R})$-structure on M is *integrable*, then each of the distributions $\mathcal{H}$ and $\mathcal{V}$ is involutive, and therefore, in a neighborhood U of each point $x \in M$, a local coordinate system $x^1, \dots, x^m$ exists such that $\mathcal{V} = \operatorname{Span}\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}$ and $\mathcal{H} = \operatorname{Span}\{\frac{\partial}{\partial x^{n+1}}, \dots, \frac{\partial}{\partial x^m}\}$.

We shall denote by h and v the projections onto $\mathcal{H}$ and $\mathcal{V}$, respectively. These operators satisfy the following relations:

$$h^2 = h; \quad v^2 = v; \quad hv = vh = 0. \tag{1.1}$$