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# VARIETIES WITH DEGENERATE GAUSS MAP WITH MULTIPLE FOCI AND TWISTED CONES

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## 0. Introduction

A smooth $n$-dimensional variety $X$ of a projective space $\mathbb{P}^N$ is called a *tangentially degenerate variety* or a *variety with degenerate Gauss map* if the rank of its Gauss map $\gamma : X \to \mathbb{G}(n, N)$ is less than $n$, $0 \leq r = \operatorname{rank} \gamma < n$. Here $x \in X$, $\gamma(x) = T_x(X)$, and $T_x(X)$ is the tangent subspace to $X$ at $x$ considered as an $n$-dimensional projective space $\mathbb{P}^n$. The number $r$ is also called the *rank* of $X$, $r = \operatorname{rank} X$. The case $r = 0$ is a trivial one: in this case, $X$ is an $n$-plane.

Let $X \subset \mathbb{P}^N$ be an $n$-dimensional smooth variety with degenerate Gauss map. Suppose that $0 < \operatorname{rank} \gamma = r < n$. Denote by $L$ a leaf of the Gauss map, $L = \gamma^{-1}(T_x) \subset X$; $\dim L = n - r = l$. The number $l$ is called the *Gauss defect* of the variety $X$ (see [1], p. 89; [2], p. 52) or *the index of relative nullity* of $X$ ([3]).

A variety with degenerate Gauss map of rank $r$ foliates into flat leaves $L$ of dimension $l$, along which the tangent subspace $T_x(X)$ is fixed. The foliation on $X$ with leaves $L$ is called the *Monge–Ampère foliation* (see, e. g., [4]–[6]).

However, unlike the traditional definition of a foliation, the leaves of the Monge–Ampère foliation have singularities. For this reason, in general, the leaves of such a foliation are not diffeomorphic to a standard leaf. In this paper, we assume that the singular points belong to the leaf $L$, and hence the leaf is an $l$-dimensional subspace of the space $\mathbb{P}^N$.

The tangent subspace $T_x(X)$ is fixed when $x$ moves along the set of regular points of $L$. For this reason, we denote it as follows: $T_L$, $L \subset T_L$. A pair $(L, T_L)$ on $X$ depends on $r$ parameters.

The varieties of rank $r < n$ are multidimensional analogues of developable surfaces of a three-dimensional Euclidean space. They were first considered by É. Cartan in connection with the study of metric deformations of hypersurfaces [7] and the study of manifolds of constant curvature [8], [9]. Recently, varieties with degenerate Gauss map of rank $r < n$ are intensively studied both from the projective point of view and from the Euclidean point of view.

An exposition of basic results concerning the geometry of varieties with degenerate Gauss map and further references can be found in [10] (Chap. 4), [11].

Griffiths and Harris ([12], § 2, p. 383–393) considered varieties with degenerate Gauss map from the point of view of algebraic geometry. Following [12], Landsberg published the book [2], which is, in some sense, a new version of [12]. Section 5 (pp. 47–50) of this book is devoted to varieties with degenerate Gauss map.

In particular, in [12], Griffiths and Harris presented a structure theorem for varieties with degenerate Gauss map. They asserted that such varieties are "built up from cones and developable varieties" and gave the proof of this assertion in the case $n = 2$. This result appeared to be complete for varieties whose Gauss maps have one-dimensional leaves. However, it is incomplete for tangentially degenerate hypersurfaces whose Gauss maps have leaves of dimension greater than one