ISSN 1066-369X, Russian Mathematics (Iz. VUZ), 2013, Vol. 57, No. 3, pp. 26–31. © Allerton Press, Inc., 2013.
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On Semirings Satisfying the Baer Criterion

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Received January 30, 2012

Abstract—It is well-known that for modules over rings the Baer injectivity criterion takes place. In this paper we prove that under one additional condition this criterion is also valid for modules over semirings. We prove that a semiring S satisfies the Baer criterion if and only if all injective (with respect to one-sided ideals of S) semimodules satisfy the above condition. We propose a new method for constructing semirings satisfying the Baer criterion.

DOI: 10.3103/S1066369X13030031

Keywords and phrases: *semiring, injective semimodule, Baer criterion.*

This work is devoted to semirings for which an analog of the Baer injectivity criterion is true. Below we assume (unless otherwise is stated) that all semirings have units, and we do not exclude the case $1 = 0$; all semimodules are assumed to be right. The notions of "subsemimodule", "semimodule homomorphism" and others used in this paper are defined analogously to the well-known notions of the theory of rings and modules; one can find these definitions, e.g., in [1].

In particular, recall that a semimodule M over a semiring S is said to be *injective* if for any S-semimodule B and any subsemimodule $A \subseteq B$ each S-homomorphism $\varphi : A \to M$ can be extended to an S-homomorphism $\overline{\varphi} : B \to M$. The semimodule M is said to be *Baer injective* if it is injective with respect to right ideals of the semiring S, i.e., for any right ideal $I \subseteq S$ every S-homomorphism $\varphi : I \to M$ can be extended to an S-homomorphism $\overline{\varphi} : S \to M$. Clearly, every injective semimodule is also Baer injective.

The following result is well-known in the theory of rings and modules (see, e.g., [2], theorem 5.7.1, P. 130, Russian transl.).

Theorem A (the Baer criterion). *A module M over a ring R is injective if and only if for any right ideal $I \subseteq R$ every R-homomorphism $\varphi : I \to M$ can be extended to an R-homomorphism $\overline{\varphi} : R \to M$.*

Therefore, a module over a ring is injective if and only if it is Baer injective. A semiring S is said to *satisfy the Baer criterion* if all Baer injective S-semimodules are injective.

In [3] (theorem 1 and the remark after it) one has shown that every commutative semigroup can be isomorphically embedded into an additive monoid of a semiring over which there exists only one Baer injective semimodule, namely, the zero semimodule. Obviously, such semirings trivially satisfy the Baer criterion, and they are not rings. One can see that this extreme "poorness" of the class of Baer injective (and hence, injective) semimodules over such semirings can be easily topped up, for instance, by adding (as a direct summand) some nonzero ring to a semiring. Then again we get a semiring satisfying the Baer criterion which is not a ring; at the same time there is a lot of injective semimodules over this semiring, all of them are modules over the added ring. In light of this, for a semiring S that satisfies the Baer criterion, the following questions arise.

Question 1. Is it true that each injective semimodule over S is a module?

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