УЧЕНЫЕ ЗАПИСКИ КАЗАНСКОГО УНИВЕРСИТЕТА
Том 154, кн. 2 Физико-математические науки *2012*

UDK 510.532

RELATIVE ENUMERABILITY AND THE *D*-C. E. DEGREES

M.M. Arslanov

Abstract

We study the relationship between the relative enumerability and the d-c. e. degrees. We prove that the degree of the halting problem is splittable into two c. e. degrees such that the upper cone of each of them contains only d-c. e. degrees which are c. e. in another one.

Key words: Turing degrees, computably enumerable degrees, relative computable enumerability, splitting, definability.

The Δ^0_2 degrees of unsolvability are basic objects of study in classical computability theory, since they are the degrees of those sets whose characteristic functions are limits of computable functions. A natural tool for understanding the Turing degrees is the introduction of hierarchies to classify various kinds of complexity. The most common such hierarchy, the arithmetical hierarchy, is itself not of much use in the classification of the Δ^0_2 degrees, since it is far too coarse. This fact has led to the introduction of hierarchies based on finer distinctions than quantifier alternation. Two such hierarchies are by now well-established. One, the *CEA hierarchy* independently defined by Arslanov [1] and Jockusch and Shore [2, 3], is, like the arithmetical hierarchy, based on the complexity of the definitions of the involved sets, replacing the alternation of quantifiers with the iteration of computable enumerability in and above a set, a significantly less powerful procedure. The second such hierarchy, the *difference hierarchy* due to Putnam [4] and Ershov [5, 6], is built up by starting with the computable enumerable sets as a base, and then iterating the operation of taking set-theoretic differences, thereby classifying sets on the basis of the difficulty of their construction in comparison with c. e. sets. Analysis of the relationship of the CEA hierarchy to the difference hierarchy is therefore a natural means of comparing the definability of sets to the inherent difficulty of their construction.

We took the first steps toward this analysis in [1]: we proved that there is a Δ^0_2 2-CEA set which is not of n-c. e. degree for any $n < \omega$. Generalizing this result Jockusch and Shore [3] proved that for any computable ordinal $\alpha < \beta$, there is a β-c. e. degree which is not α-CEA, while, on the other hand, that for every uniformly given class of Δ^0_2 degrees, there is a Δ^0_2 2-CEA degree which is not in this class. From the latter result, it follows that for each $\alpha \le \omega$, there is a 2-CEA set which is not of α-c. e. degree. This result is more interesting when $\alpha > 1$, since d-c. e. sets, and hence 2-CEA sets, not of c. e. degree had already been constructed by Cooper (unpublished). In [7] we took a further step towards analyzing the relationship of this second level of the CEA hierarchy to the difference proving that any ω-c. e. degree which is 2-CEA is also 2-c. e. In this paper we also obtained a second result which imposes a significant limit on possible extensions of this result: there exists a d-c. e. set C such that for every $n \ge 3$, there exists a set A which is simultaneously C-CEA and $(n+1)$-c. e., yet fails to be of n-c. e. degree.

Further results in this direction are obtained in [8]. Let $\mathbf{u}$ and $\mathbf{v}$ be c. e. degrees such that $\mathbf{v} < \mathbf{u}$. Then there is a d-c. e. degree $\mathbf{d}$ such that $\mathbf{v} < \mathbf{d} < \mathbf{u}$ and $\mathbf{d}$ is not c. e. in $\mathbf{v}$.

This result naturally raises the following problem, which has a long history. Let $\mathbf{a} < \mathbf{b}$ be non-computable c. e. degrees. Is there a CEA in $\mathbf{a}$ d-c. e. degree $\mathbf{d} < \mathbf{b}$ such that $\mathbf{b}$ is not c. e.?

Below we list all so far known results on this question:

1. [9] Let $\mathbf{a}$ be a non-computable c. e. degree such that $\mathbf{a}' = \mathbf{0}'$. Then there is a non-c. e. but c. e. in $\mathbf{a}$ degree $\mathbf{b} > \mathbf{a}$. Moreover,

2. [10] Let $\mathbf{c} < \mathbf{h}$ be c. e. degrees such that $\mathbf{c}$ is low and $\mathbf{h}$ is high. Then there is a degree $\mathbf{a} < \mathbf{h}$ such that $\mathbf{a}$ is CEA in $\mathbf{c}$ degree.

3. [10] For all high c. e. degrees $\mathbf{h} < \mathbf{g}$, there is a properly d-c. e. degree $\mathbf{a}$ such that $\mathbf{h} < \mathbf{a} < \mathbf{g}$ and $\mathbf{a}$ c. e. in $\mathbf{h}$.

4. [10] There is a c. e. degree $\mathbf{a}, \mathbf{0} < \mathbf{a} < \mathbf{0}'$ such that for any c. e. in $\mathbf{a}$ degree $\mathbf{b} > \mathbf{a}$, if $\mathbf{b} \leqslant \mathbf{0}'$ then $\mathbf{b}$ is c. e.

5. [11] Let $\mathbf{a} > \mathbf{0}$ be a superlow degree. Then there is a properly d-c. e. degree $\mathbf{d} > \mathbf{a}$ such that $\mathbf{d}$ is c. e. in $\mathbf{a}$.

   (A set A is called *superlow* if $A' \equiv_{tt} \emptyset'$. A degree is superlow if it contains a superlow set.)

(A set A is called *superlow* if $A' \equiv_{tt} \emptyset'$. A degree is superlow if it contains a superlow set.)

These results allow us to formulate the following well-known hypothesis which is still can be considered as an open problem.

Conjecture. For every low c. e. degree $\mathbf{a} > \mathbf{0}$, there is a CEA in $\mathbf{a}$ d-c. e. degree $\mathbf{b}$ which is not c. e.

Recent investigations have shown that this problem is closely related to another important problem on definability of the c. e. degrees in the d-c. e. degree structure, and in this investigation deeper understanding of the splitting properties of d-c. e. degrees may be useful.

Let $\mathbf{a} > \mathbf{0}$ be a properly d-c. e. degree and let $\mathbf{b}$ be a c. e. degree such that $\mathbf{b} < \mathbf{a}$. Since $\mathbf{a}$ is c. e. in some c. e. degree $\mathbf{a_0} < \mathbf{a}$, it follows from the Sacks Splitting Theorem, relativized to $\mathbf{a_0} \cup \mathbf{b} < \mathbf{a}$, that $\mathbf{a}$ is splittable into two 2-CEA degrees which are above $\mathbf{b}$, i.e. there are 2-CEA-degrees $\mathbf{c}_0$ and $\mathbf{c}_1$ such that $\mathbf{c}_0 \cup \mathbf{c}_1 = \mathbf{a}$ and $\mathbf{b} < \mathbf{c}_0 < \mathbf{a}$, $\mathbf{b} < \mathbf{c}_1 < \mathbf{a}$. Moreover, Arslanov, Cooper and Li [12, 13] proved that any c. e. degree is splittable in the d-c. e. degrees over any *low* d-c. e. degree. In this paper we prove that $\mathbf{0}'$ is splittable into c. e. degrees $\mathbf{v}_0$ and $\mathbf{v}_1$ such that for every d-c. e. degree $\mathbf{d}$ and each $i \leq 1$, if $\mathbf{v}_i \leq \mathbf{d}$ then $\mathbf{d}$ is c. e. in $\mathbf{v}_{1-i}$.

We adopt the usual notational conventions found, for instance, in [14]. In particular, we write $[s]$ after functionals and formulas to indicate that every functional or parameter therein is evaluated at stage s. In particular, for an oracle X and c. e. functional Φ, $\Phi(X; y, s)$ means only that at most s steps are allowed for the computation from oracle X to converge, whereas $\Phi(X; y)[s]$ means also that the approximation X_s is used as the oracle, and may mean as well that some function-value $y(s)$ is being used as the argument for the computation. When using a c. e. oracle, we adopt the common practice of taking the use function to be nondecreasing in the stage.

Theorem 1. *There exists a splitting of* $\mathbf{0}'$ *into c. e. degrees* $\mathbf{v}_0$ *and* $\mathbf{v}_1$ *such that for every* d*-c. e. degree* $\mathbf{d}$ *and each* $i \leq 1$, *if* $\mathbf{v}_i \leq \mathbf{d}$ *then* $\mathbf{d}$ *is c. e. in* $\mathbf{v}_{1-i}$.

Proof. We will construct c. e. sets V_0 and V_1 so that the degrees $\mathbf{v}_i = \deg V_i$ have the desired properties. We also construct auxiliary c. e. sets U_0, U_1.

This is ensured by the following two types of requirements.

To ensure that $\emptyset' \not\leq_T V_i$, we satisfy requirements

- P^i_e: $U_i \neq \Theta^{V_i}_{i,e}$ (for each partial computable functional $\Theta_{i,e}$).

To ensure that for all d-c. e. sets D, if $V_i \leq_T D$ then D is of degree c. e. in V_{1-i}, we satisfy requirements.

- R^i_e: $D_e = \Delta^{V_i \oplus Q^i_e}_{i,e}$ & $Q^i_e \leq_T V_i \oplus D_e$ (for each d-c. e. set D_e we build an associated d-c. e. set Q^i_e c. e. in V_{1-i} and a partial computable functional $\Delta_{i,e}$).

  The condition $Q^i_e \leq_T V \oplus D_e$ will be met by the usual permitting argument.

To ensure that Q^i_e is c. e. in V_{1-i} we use a common method which works as follows. When an integer x is enumerated into Q^i_e at stage s we appoint a certain marker $\alpha(x)$. Then we allow x to be removed from Q^i_e at a later stage t only if $V_{1-i} \upharpoonright \alpha(x)[t-1] \neq$ $\neq V_{1-i} \upharpoonright \alpha(x)[t]$.

The condition $\emptyset' \leq_T V_0 \oplus V_1$ will follow from the construction directly.

The basic strategy for P-requirements in isolation is the one developed by Friedberg and Muchnik:

(1) Pick an unused witness x from the column associated with this requirement ($< i, \omega >$ for $i \leq 1$), which is larger than all higher-priority restraints, and keep it out of U.

(2) Wait for $\Theta^V_e(x) \downarrow = 0$.

(3) Put x into U and protect $V \upharpoonright (\theta(x)+1)$.

The basic strategy for R-requirements in isolation is to build $\Delta^{V \oplus Q}$, ensuring that it is total and computes D correctly. Since we build the set V during the construction, we may easily meet this requirement by changing V, if necessary.

While the strategies for the requirements in isolation are thus very simple, there are obviously several conflicts between them. The R-strategy threatens to contribute into V infinitely many numbers while P-restraints of lower priority may obstruct it.

Before giving the explicit construction we first explain the intuition for the P_i- and S_j-requirements below one R_e-strategy.

Basic module for the R^i_e*-strategy above* P*-requirements.*

We use an ω-sequence of "cycles", where each cycle k proceeds as follows:

(1) At a stage s set $\Delta^{V_i \oplus Q^i_e}_{i,e}(k) = D_{e,s}(k)$ with a *use* $\delta_{i,e,s}(k) >$ all P^i- and P^{1-i}-restraints, and $\delta_{i,e,s}(k) > \delta_{i,e,s}(k-1)$ and start cycle $k+1$ to run simultaneously with cycle k.

(2) Wait for $D_e(k)$ to change (at a stage t, say).

(3) (i) Enumerate $\delta_{i,e,s}(k)$ into Q^i_e,

(ii) set $\Delta^{V^i \oplus Q^i_e}_{i,e}(k) = D_{e,t}(k)$ with a new *use* $\delta_{i,e,t}(k) >$ all P^i-restraints, and $\delta_{i,e,t}(k) > \delta_{i,e,s}(k)$, and

(iii) appoint the marker $\alpha_i(\delta_{i,e,s}(k))$ as the first integer y such that $y \geq \delta_{i,e,t}(k)$ and $y = < 2, l >$ for some l.

(4) Wait for $D_e(k)$ to change back (at a stage u, say).

(5) We need

– to keep Q^i_e below $V^i \oplus D_e$ (at stage t k enters D_e, and we put $\delta_{i,e,s}(k)$ into Q^i_e. Now k leaves D_e).

– to correct the axiom $\Delta_{i,e}^{(V^i \oplus Q^i_e)}(k) = D_e(k)$

We have two possibilities to achieve this:

– either by enumerating $\delta_{i,e,s}(k)$ into V^i
– or by removing $\delta_{i,e,s}(k)$ from Q^i_e (in this case we need to enumerate $\alpha_i(\delta_{i,e,s}(k))$ into V_{1-i}).

The crucial point here is that our choice between these two possibilities depends upon the priority ordering of requirements P^i and P^{1-i} that may be injured:

a) If the highest-priority strategy which would be injured by this correction is a P^i-strategy (or there is no strategy at all that would be injured), then enumerate $\alpha_i(\delta_{i,e,s}(k))$ into V_{1-i} and remove $\delta_{i,e,s}(k)$ from Q^i_e.

b) Otherwise, put $\delta_{i,e,s}(k)$ into V^i, and set $\Delta_{i,e}^{V^i \oplus Q^i_e}(k) = D_{e,u}(k)$.

Set $\Delta_{i,e}^{V_i \oplus Q^i_e}(k) = D_{e,u}(k)$ with the same *use* $\delta_{i,e,u}(k) = \delta_{i,e,t}(k)$.

In both cases start cycle $k+1$ to run simultaneously.

We now give the construction. We say that the axiom $\Delta_{i,e}^{V^i \oplus Q^i_e}(k) = D_e(k)$ *requires correction* at stage s if at a stage $t < s$ we set $\Delta_{i,e}^{V^i \oplus Q^i_e}(k) = D_{e,t}(k)$ with a *use* $\delta_{i,e,t}(k), D_{e,s}(k) \neq D_{e,t}(k)$, and $(V^i \oplus Q^i_e)_t \upharpoonright \delta_{i,e,t}(k) = (V^i \oplus Q^i_e)_s \upharpoonright \delta_{i,e,t}(k)$.

Stage $s = 0$. Set $U_0 = V_0 = V_1 = \emptyset[0]$. $x^0_{e,0} =< 0, e >$, $x^1_{e,0} =< 1, e >$.

Stage $s > 0$. Fix e such that $s = \langle e, m\rangle$ for some m.

Substage 1 (P^0_e-requirement).

a) If $\Theta_{0,e}^{V_0}(x^0_e)[s] \downarrow = 0$ and $x^0_{e,s-1} \notin U_{0,s}$, then enumerate $x^0_{e,s-1}$ into $U_{0,s}$, and protect $V_0 \upharpoonright \theta_{0,e,s}(x^0_{e,s-1})$ with priority P^0_e.

b) If $\Theta_{0,e}^{V_0}(x^0_e)[s] \downarrow = U_{0,s}(x^0_{e,s-1}) = 1$, then define

$$x^0_{e,s} = (\mu x)[(\exists y)(\forall j)(x = \langle 0, y\rangle \wedge x > \text{all } R^i_j\text{-uses assigned so far}].$$

Otherwise, set $x^0_{e,s} = x^0_{e,s-1}$.

Substage 2 (P^1_e-requirement). Similar to the previous case with necessary changes ($\Theta^0_e, V_0, U_0, x^0_e, \theta_{0,e}$ by $\Theta^1_e, V_1, U_1, x^1_e, \theta_{1,e}$ accordingly).

Substage 3. Let z be the greatest integer such that for any $k < z$ there exists a stage $s' < s$ such that at stage s' the axiom $\Delta_{i,e}^{V_i \oplus Q_(i,e)}(k) = D_e(k)[s']$ was set. Let $k < z$ be the smallest integer (if any) such that the axiom $\Delta_{i,e}^{(V_i \oplus Q_{i,e})}(k) = D_e(k)$ requires correction at stage s. Let t be a stage at which the axiom $\Delta_{i,e}^{(V_i \oplus Q_{i,e})}(k) = = D_e(k)$ was set.

We consider two cases.

Case 1) $D_{e,s}(k) = 1$. In this case we proceed as in step (3) of the Basic Module:
(i) enumerate $\delta_{i,e,t}(k)$ into $Q_{i,e}$,

(ii) set $\Delta_{i,e}^{V_i \oplus Q_{i,e}}(k) = D_{e,s}(k)$ with a new *use* $\delta_{i,e,s}(k) >$ all P-restraints, and $\delta_{i,e,s}(k) > \delta_{i,e,t}(k)$, and

(iii) appoint the marker $\alpha_i(\delta_{i,e,s}(k))$ as the first integer y such that $y \geq \delta_{i,e,s}(k)$ and $y =< 2, l >$ for some l.

Case 2) $D_{e,s}(k) = 0$. Therefore, there is a stage $u < s$ such that $D_{e,u}(k) = 1$, and at stage u we (re)set the axiom $\Delta_{i,e}^{V_i \oplus Q_{i,e}}(k) = D_e(k)$. It follows also that at stage u we enumerated $\delta_{p,e,t}(k)$ into $Q_{i,e}$. In this case we proceed as in step (5) of the Basic Module:

a) if the highest-priority strategy which would be injured by the $Q_{i,e}(\delta_{i,e,t}(k))-$ or $V_i(\delta_{i,e,t}(k))-$ correction is a P-strategy (or there is no strategy at all that would be injured), then enumerate $\alpha_i(\delta_{i,e,t}(k))$ into V_{1-i} and remove $\delta_{i,e,t}(k)$ from $Q_{i,e}$.

b) Otherwise, put $\delta_{i,e,t}(k)$ into V_i.

Set $\Delta_{i,e}^{V_i \oplus Q_{i,e}}(k) = D_{e,s}(k)$.

Substage 4. If none of the axioms $\Delta_{i,e}^{V_i \oplus Q_{i,e}}(k) = D_e(k)$ for $k < z$ requires correction at stage s, then set the new axiom $D_{e,s}(z) = \Delta_{i,e}^{V_i \oplus Q_{i,e}}(z)$ with a *use* $\delta_{i,e,s}(z) >$ all P-, R-restraints.

Substage 5. Go to stage $s + 1$.

End of the construction.

Verification.
Let $\mathbf{v_i}$=deg(V_i), $i \leq 1$.

Lemma 1. $Q_{i,e} \leq_T V_i \oplus D_e$.

Proof. To $V_i \oplus D_e$-computably compute whether $x \in Q_{i,e}$, first find a stage u at which a new axiom $D_e(y) = \Delta_{i,e}^{V_i \oplus Q_{i,e}}(y)$ with a *use* $\delta_{i,e,u}(y) \geq x$ is settled. Obviously, such a stage u exists.

Suppose now that $x = \delta_{i,e,s}(k)$ was chosen as a *use* for some $\Delta_{i,e}^{V_i \oplus Q_{i,e}}(k)$ at a stage $s \leq u$ (otherwise, $x \notin Q_{i,e}$). Find a stage $v \geq u$ at which $V_{i,v} \upharpoonright x = V_i \upharpoonright x$ and $D_{e,v}(x) = D_e(x)$. Now $x \in Q_{i,e}$ if and only if $x \in Q_{i,e,v}$.

□

Lemma 2. *If* $D_e = \Phi_e^{U \oplus V}$ *then* $D_e \leq_T V \oplus Q_e$.

Proof. It follows immediately by construction. □

Lemma 3. $Q_{i,e}$ *is c. e. in* V_{1-i}.

Proof. It follows immediately from the construction. □

Lemma 4. *For each* $i \leq 1$ *and* $e \in \omega$, *requirements* P_e^i *are eventually satisfied.*

Proof. Fix e and assume by induction that the Lemma holds for all $j < e$. Choose s minimal so that no P_j^i-restraints may be injured by some R-requirement. By construction we may injure P_e^i by finitely many times contributing some integers into V_i to protect the V_{1-i}-restraint of higher priority. But beginning at some stage s we take witnesses for Δ-*uses* greater than the V_{1-i}-restraints, after that we meet the requirement P_e^i.

□

Lemma 5. $\mathbf{0}' = \mathbf{v_0} \cup \mathbf{v_1}$.

Proof. Suppose, in contrary, that $\mathbf{v_0} \cup \mathbf{v_1} < \mathbf{0}'$. Then by [8, Theorem 4.1] there exists a d-c. e. degree $\mathbf{d}$ such that $\mathbf{v_0} \cup \mathbf{v_1} < \mathbf{d}$ and $\mathbf{d}$ is not c. e. in $\mathbf{v_0} \cup \mathbf{v_1}$, and therefore it is not c. e. in $\mathbf{v}_1$. We have $\mathbf{v}_0 < \mathbf{d}$ and $\mathbf{d}$ is not c. e. in $\mathbf{v_1}$, a contradiction. □

□

The author was partially supported by the Russian Foundation for Basic Research (Grant No. 10-01-00399-a) and the Presidential Program for Support of Leading Scientific Schools (Grant No. 5383.2012.1).

Резюме

М.М. Арсланов. Относительная перечислимость и d-вычислимо перечислимые степени.

В работе изучается связь между относительной перечислимостью и d-вычислимо перечислимыми степенями. Доказано, что тьюринговая степень проблемы остановки машины Тьюринга разложима на такие две вычислимо перечислимые степени, что верхний конус каждой из них состоит только из тех d-вычислимо перечислимых степеней, которые перечислимы относительно второй степени.

Ключевые слова: тьюринговые степени, вычислимо перечислимые степени, относительная перечислимость, разложение, определимость.

References

1. *Arslanov M.M.* On a hierarchy of the degrees of unsolvability // Ver. Metody i Kibernetika. – 1982. – No 18. – P. 10–17 (in Russian).
2. *Jockusch C.G., Jr., Shore R.A.* Pseudo jump operators I: The R.E. case // Trans. Amer. Math. Soc. – 1983. – V. 275, No 2. – P. 599–609.
3. *Jockusch C.G., Jr., Shore R.A.* Pseudo jump operators II: Transfinite iterations, hierarchies, and minimal covers // J. Symb. Logic. – 1984. – V. 49, No 4. – P. 1205–1236.
4. *Putnam H.* Trial and error predicates and the solution to a problem of Mostowski // J. Symb. Logic. – 1965. – V. 30, No 1. – P. 49–57.
5. *Ershov Y.* On a hierarchy of sets I // Algebra i Logika. – 1968. – V. 7, No 1. – P. 47–73 (in Russian).
6. *Ershov Y.* On a hierarchy of sets II // Algebra i Logika. – 1968. – V. 7, No 4. – P. 15–47 (in Russian).
7. *Arslanov M.M., LaForte G.L., Slaman T.A.* Relative enumerability in the difference hierarchy // J. Symb. Logic. – 1998. – V. 63, No 2. – P. 411–420.
8. *Arslanov M.M., Lempp L., Shore R.A.* On isolating c. e. and isolated d-c. e. degrees // S.B. Cooper, T.A. Slaman, S.S. Wainer (Eds.) Computability, enumerability, unsolvability (London Math. Soc. Lect. Note Series). – Cambridge: Cambridge Univ. Press, 1996. – No 224. – P. 61–80.
9. *Soare R.I., Stob M.* Relative recursive enumerability // Proc. Herbrand Symposium, Logic Colloquium. – 1981. – P. 299–324.
10. *Arslanov M.M., Lempp S., Shore R.A.* Interpolating d-r.e. and REA degrees between r.e. degrees // Ann. Pure Appl. Logic. – 1995. – V. 78. – P. 29–56.

11. *Arslanov M.M., Cooper S.B., Li A.* There is no low maximal d.c.e. degree – Corrigendum // Math. Logic Quart. – 2004. – V. 50, No 6. – P. 628–636.

12. *Arslanov M.M.* The Ershov Hierarchy // Cooper S.B., Sorbi A. (Eds.) Computability in Context: Computation and Logic in the Real World. – Imperial College Press/World Scientific, 2011. – P. 49–100.

13. *Arslanov M.M., Cooper S.B., Li A.* There is no low maximal d.c.e. degree // Math. Logic Quart. – 2000. – No 46. – P. 409–416.

14. *Soare R.I.* Recursively Enumerable Sets and Degrees: A Study of Computable Functions and Computably Generated Sets (Perspectives in Mathematical Logic). – Berlin: Springer-Verlag, 1987. – 437 p.

Поступила в редакцию
15.12.10

Arslanov, Marat Mirzaevich – Doctor of Physics and Mathematics, Professor, Head of the Department of Algebra and Mathematical Logic, Kazan Federal University, Kazan, Russia.

Арсланов Марат Мирзаевич – доктор физико-математических наук, профессор, заведующий кафедрой алгебры и математической логики Казанского (Приволжского) федерального университета.

E-mail: *Marat.Arslanov@ksu.ru*